ELEMENTARY VECTOR ANALYSIS

WITH APPLICATION TO
GEOMETRY AND PHYSICS

BY

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TO

JOHN HENRY MICHELL, F.R.S.

WHOSE HELP AND ADVICE

HAVE BEEN AT ALL TIMES MOST WILLINGLY GIVEN,

AND TO WHOSE INSPIRATION

THE WRITING OF THE FOLLOWING PAGES WAS LARGELY DUE,

THIS BOOK IS GRATEFULLY ASCRIBED.

PREFACE

My object in writing this book was to provide a simple exposition of elementary Vector Analysis, and to show how it may be employed with advantage in Geometry and Mechanics. It was thought unnecessary, in the present volume, to enter upon the more advanced parts of the subject, built upon the ideas of gradient, curl and divergence. Vector algebra and the differentiation of vectors with respect to one scalar variable furnish a powerful instrument even for the higher parts of dynamics.

The work does not claim to be a complete text-book in either Geometry or Mechanics, though a good deal of ground is covered in both subjects. The use of vector analysis in the former is abundantly illustrated by the treatment of the straight line, the plane, the sphere and the twisted curve, which are dealt with as fully as in most elementary books, and a good deal more concisely. In Mechanics I have explained and proved all the important elementary principles. The equations of equilibrium for a rigid body are deduced from the equations of motion. This is contrary to the ordinary practice and, of course, is not recommended for young beginners. But for a student who is able to read this volume, it is certainly desirable to show that the principles of statics are only particular cases of the dynamical ones, and that the long line of argument followed by text-books in Statics, to prove the theorems about moments, parallel forces, couples and the equilibrium of bodies, is really unnecessary. All these theorems are immediately deducible from the equations of motion of a rigid body, as shown in Chapter VIII.

Another departure from the ordinary practice has been made in connection with the theory of *centroids*. Most students gain

their introduction to centroids through centre of gravity. But at a later stage they should learn that centre of gravity is only a particular case of centroids, and that a presentation of the subject may be given which includes all cases. Arts. 9-11 were written with this object in view. It is because most students regard centre of gravity as the very essence of centroid that we continually meet such expressions as "centre of gravity of an area" or "centre of gravity of a cross section." Centroids of area and volume exist in their own right, and are quite independent of mass and weight.

In treating the geometry of the straight line, plane and sphere, my object is primarily to explain the vector method and notation, and *not* to show their superiority over other methods. The reader must decide for himself which is preferable. In connection with twisted curves, the use of vectors seems decidedly advantageous. In the case of the plane and the sphere it is chiefly brevity of expression that is gained; though a comparison of the geometrical work in Chapters III. and IV. with the corresponding theory given in books on analytical geometry will be instructive to the reader. *Vectors were, however, not designed for use in elementary plane geometry*; and any two-dimensional geometry in the sequel is introduced merely by way of easy illustration of the vector method. Vector analysis is intended essentially for three-dimensional calculations; and its greatest service is rendered in the domains of mechanics and mathematical physics.

After much consideration I decided to employ the dot and cross notation for products of vectors. This has always appeared to me the most convenient, particularly for the treatment of the linear vector function.

During the preparation of this book I have been greatly indebted to Mr. J. H. Michell, M.A., F.R.S., who read the MS. and proof sheets, and made many valuable suggestions that have been incorporated in the work. I was allowed free use of his honour lectures in Mixed Mathematics, Part I., at Melbourne University; and it was his manner of treating the subject that first led me to undertake the study of Vector Analysis. His interest and encouragement in the writing of this book have been largely responsible for its final appearance.

My thanks are also due to Prof. W. P. Milne, the editor of this series, who also read the MS. and made many excellent suggestions which I was glad to adopt. Acting on his advice, I added the Historical Introduction, which will prove interesting to many readers. I am also indebted to Mr. D. K. Picken, M.A., Master of Ormond College, for certain introductory ideas, which have to some extent influenced my presentation of the subject. And I take this opportunity of thanking Mr. Ian W. Wark, B.Sc., of Ormond College, who generously undertook the task of verifying the exercises to each chapter, and furnishing answers where necessary. I am also grateful to my college friend, Dr. T. M. MacRobert, of Glasgow University, who kindly offered to revise the final proofs.

As to other literature, I have elsewhere * acknowledged my great indebtedness to E. B. Wilson's *Vector Analysis*, which was my early instructor in the subject; and during the writing of the following pages I was influenced both consciously and unconsciously by Professor Wilson's book. Coffin's *Vector Analysis*, another American book, has also been frequently consulted by the author.

In conclusion, I wish to thank the Publishers for their unfailing courtesy, and the Printers for the excellence of their work.

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UNIVERSITY OF MELBOURNE,
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* "A plea for a more general use of Vector Analysis in Applied Mathematics," *Math. Gazette*, Jan. 1917.

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HISTORICAL INTRODUCTION *

THE method of subjecting vector quantities to scalar algebra by resolution into three components is due to the French philosopher Descartes (1596-1650). The need of a calculus for operating directly on vectors has long been recognised; and in 1679 Leibnitz made an attempt at meeting the need, but with little success. The problem attracted the attention of subsequent writers, and in 1806 Argand showed how a geometrical representation could be given to the complex number. This representation of unreal quantities by coplanar vectors has proved of considerable importance in the theory of complex variables; but at the same time it gave the unfortunate impression that the theory of real vectors is necessarily dependent on that of complex numbers—an impression which has not even yet entirely disappeared. A little later, in 1826, appeared the *Barycentrisches Calcul* by Möbius, one of the best known of Gauss's pupils. This work is a forerunner of the more general analysis of geometric forms subsequently developed by Grassmann. The *Calcolo delle Equipollenze* devised by Bellavitis in 1832, and subsequently enlarged, actually deals systematically with the geometric addition of vectors and the equality of vectors.

The years 1843-44 are remarkable in the history of mathematics for the almost simultaneous appearance of Hamilton's *Quaternions* and the *Ausdehnungslehre* of Grassmann. **William Rowan Hamilton** was born at Dublin on 4th August, 1805. His father, Archibald Hamilton, had migrated from Scotland in his youth.

* This Introduction is not essential to the argument of the book. The author hopes, however, that it will add to the value and general interest of the work.

The son gave early evidence of genius, being a remarkable linguist and displaying great mathematical talent. He entered Trinity College, Dublin, in 1824, where he had a brilliant and unprecedented career. His ability was so conspicuous that in 1827, while still an undergraduate, he was asked to apply for the vacant Andrews' Professorship of Astronomy in the University of Dublin, and was appointed to the position. He was not specially qualified as a practical astronomer; but the conditions of his appointment allowed him to advance the cause of Science in the way he felt best able to do so. In 1835, while acting as secretary to the B.A.A.S. at its meeting in Dublin, he received a knighthood; and two years later the importance of his scientific work was recognised by his election as President of the Royal Irish Academy. His mathematical work continued uninterrupted till his death on 2nd September, 1865, at the age of sixty.

It often happens that we get our most important ideas while not formally working at a subject, perhaps while walking in the country or by the sea, or even in more commonplace surroundings. From a letter of Hamilton's we learn that, on 16th October, 1843, while he was walking beside the Royal Canal on his way to preside at a meeting of the Academy, the thought flashed into his mind which gave the key to a problem that had been occupying his thoughts, and led to the birth and development of the subject of Quaternions. He announced the discovery at that meeting of the Academy, and asked permission to read a paper on quaternions at the next, which he did on 13th November. During the next few years he expanded the subject, and published his *Lectures on Quaternions* in 1853, while the *Elements of Quaternions* appeared in 1866, soon after his death.

In August, 1844, appeared the first edition of Grassmann's *Lineale Ausdehnungslehre*, a treatise of over 300 pages. Hermann Günther Grassmann was born at Stettin on 15th April, 1809, and died at the same place in 1877. He held the post of instructor in mathematics and science at a gymnasium in his native town. The systems of Hamilton and Grassmann may be regarded as the parents of modern Vector Analysis. The two authors,

working independently and along different lines, each developed a wonderful analysis. The quaternion is a sort of "sum" or complex of a scalar and a vector, though originally defined as the "quotient" of two vectors. The *Ausdehnungslehre* is an algebra of geometric forms. Both systems contain an algebra of vectors, and both, as finally developed, a theory of linear vector functions. In Hamilton's we have also the linear quaternion function, and in Grassmann's the linear function applied to the quantities of his algebra of points. "Grassmann's algebra of points may be regarded as the application of the methods of multiple algebra to the notions connected with tetrahedral coordinates, just as his or Hamilton's algebra of vectors may be regarded as the application of the methods of multiple algebra to the notions connected with rectilinear coordinates." *

Each of the above systems is a remarkable and potent instrument of analysis; and the devotees of each have faithfully striven to prove its power and utility in the various branches of mathematics. Among Hamilton's disciples the most noted was Prof. P. G. Tait. **Peter Guthrie Tait** was born at Dalkeith, Scotland, on 28th April, 1831. He was educated at the Edinburgh Academy, and then for one session (1847) at the Edinburgh University. The following year he proceeded to Cambridge, where he entered Peterhouse before his eighteenth birthday. He became Senior Wrangler, and in 1852 was elected Fellow and Lecturer of Peterhouse, where he remained for two years longer. At the end of that time he was appointed Professor of Mathematics at Queen's College, Belfast. Here he was introduced to Hamilton and Quaternions, and became a staunch friend of both. In 1860 he was appointed to the Professorship of Natural Philosophy in the University of Edinburgh, a position which he held till his death in 1901. His *Elementary Treatise on Quaternions* was published in 1867, and a second edition in 1873.

However, neither the system of Hamilton nor that of Grassmann met the needs of physicists or applied mathematicians, being too general and too complex for the requirements of ordinary

* Gibbs, "Quaternions and the *Ausdehnungslehre*," *Nature*, vol. 44, pp. 79-82 (1891).

calculations. The ideas involved in the scalar and vector quantities of mechanics and physics are much simpler than those of Hamilton's theory, in which imaginaries play a large part, and vectors and scalars appear as degenerate quaternions rather than in their own right. The feeling became general that a system was needed in which the ideas were more simply related to the facts of nature. Mathematicians in various countries therefore began to adapt the results of Hamilton and Grassmann to the more elementary requirements; and although these investigations were carried on independently and from different points of view, the analyses arrived at are identical as regards the elements and functions introduced. It is mainly in notation and terminology that the differences lie. In Germany the starting point was the *Ausdehnungslehre*; and among those who contributed to the formation and adoption of a simpler analysis may be mentioned Föppl, Abraham, Bucherer, Fischer, Ignatowsky and Gans. In England, Heaviside deserves special mention; while in America, Prof. W. Gibbs did much admirable work.

Josiah Willard Gibbs was born at New Haven, Connecticut, on 11th February, 1839. His father, who bore the same name, was Professor of Sacred Literature in the Yale Divinity School from 1824 till 1861. The son entered Yale in 1854, and graduated four years later after a distinguished career. He continued his studies at New Haven, and in 1863 received the degree of Ph.D., being then appointed Tutor at Yale for a period of three years. At the end of this term he visited Europe, studying at Paris during the winter of 1866-67, and at Berlin and Heidelberg during the ensuing two years. He returned to New Haven in June, 1869, and two years later was appointed to the Professorship of Mathematical Physics at Yale, a position which he held till his death on 28th April, 1903. As an investigator in Mathematical Physics, Gibbs soon gave evidence of his powers by the publication of several papers in Thermodynamics, among which the well-known memoir *On the Equilibrium of Heterogeneous Substances* has proved of fundamental importance to Physical Chemistry. In the Electromagnetic Theory of Light also, Gibbs did much work of permanent value; and

learned societies and Universities, both in Europe and in America, formally recognised the merits of his contributions to science.

In lecturing to students Prof. Gibbs felt the need of a simpler form of Vector Analysis than was then available. Being familiar with the work of both Hamilton and Grassmann, he was able to adapt to his requirements the best and simplest parts of both systems, thus developing an analysis which he used freely in his University teaching. In 1881 and 1884 he printed at New Haven, privately for the use of his pupils, a pamphlet entitled *Elements of Vector Analysis*, giving a concise account of his system. This pamphlet was to some extent circulated also among others specially interested in the subject. It was not till twenty years later that Prof. Gibbs reluctantly consented to the formal publication, in a fairly complete form, of the vector analysis to which he was led.

"The reluctance of Professor Gibbs to publish his system of vector analysis certainly did not arise from any doubt in his own mind as to its utility, or the desirability of its being more widely employed: it seemed rather to be due to the feeling that it was not an original contribution to mathematics, but was an adaptation, for special purposes, of the work of others. Of many portions of the work this is of course necessarily true; and it is rather by the selection of methods and by systematization of the presentation that the author has served the cause of vector analysis. But in the treatment of the linear vector function and the theory of dyadics to which this leads, a distinct advance was made which was of consequence not only in the more restricted field of vector analysis, but also in the broader theory of multiple algebra in general." *

Meanwhile in England **Oliver Heaviside** was engaged in a similar task. His work in the Electromagnetic Theory led him first to study quaternions as probably what he needed to simplify the analysis, and then to reject them as totally unsuitable. In adapting the results of Hamilton and Tait to his own requirements he arrived at a vector algebra practically identical with that of Gibbs. The difference of notation was of course to be expected.

* P. xix of the Biographical Sketch by H. A. Bumstead in Gibbs's *Scientific Papers*.

Heaviside adhering partly to that employed by the quaternionists, but introducing the admirable practice of representing vectors by Clarendon symbols. On receiving a copy of Gibbs's New Haven pamphlet, Heaviside expressed his warm admiration and approval, though still preferring his own notation.

The work of Gibbs and Heaviside drew forth denunciations from Prof. Tait, who considered any departure from quaternionic usage in the treatment of vectors to be an enormity. "Even Prof. Gibbs," he wrote,* "must be ranked as one of the retarders of quaternion progress, in virtue of his pamphlet on *Vector Analysis*, a sort of hermaphrodite monster compounded of the notations of Hamilton and of Grassmann." Prof. Gibbs was well able to look after himself, and in his reply † had a long way the best of the argument. He wisely separates the quaternionic question from that of a suitable notation, and argues powerfully against the treatment of vectors by quaternions. The discussion thus begun, continued for some years with Tait and other quaternionists ranged on one side, and Gibbs and Heaviside on the other. The contributions of Heaviside add a human touch to the controversy, and make very interesting reading even at the present day.

"'Quaternion' was, I think, defined by an American schoolgirl to be 'an ancient religious ceremony.' This was, however, a complete mistake. The ancients—unlike Prof. Tait—knew not and did not worship Quaternions." ‡

"It is known that Sir W. Rowan Hamilton discovered or invented a remarkable system of mathematics, and that since his death the quaternionic mantle has adorned the shoulders of Prof. Tait, who has repeatedly advocated the claims of Quaternions. Prof. Tait in particular emphasises its great power, simplicity, and perfect naturalness, on the one hand; and on the other tells the physicist that it is exactly what he needs for his physical purposes. It is also known that physicists, with great obstinacy, have been careful (generally speaking) to have nothing to do with Quaternions; and, what is equally

* Preface to the third edition of *Quaternions*.

† *Nature*, vol. 43, pp. 511-13 (1891).

‡ *Electromagnetic Theory*, vol. 1, p. 136 (London, 1893).

remarkable, writers who take up the subject of Vectors are (generally speaking) possessed of the idea that Quaternions is not exactly what they want, and so they go tinkering at it, trying to make it a little more intelligible, very much to the disgust of Prof. Tait, who would preserve the quaternionic stream pure and undefiled. Now, is Prof. Tait right, or are the defilers right? Opinions may differ. My own is that the answer all depends upon the point of view. If we put aside practical applications to Physics, and look upon Quaternions entirely from the quaternionic point of view, then Prof. Tait is right, thoroughly right, and Quaternions furnishes a uniquely simple and natural way of treating *quaternions*. Observe the emphasis." *

"But when Prof. Tait vaunts the perfect fitness and naturalness of quaternions for use by the physicist in his enquiries, I think that he is quite wrong. For there are some very serious drawbacks connected with quaternions, when applied to vectors. The quaternion is regarded as a complex of scalar and vector, and as the principles are made to suit the quaternion, the vector itself becomes a degraded quaternion, and behaves as a quaternion. That is, in a given equation, one vector may be a vector, and another be a quaternion. Or the same vector in one and the same equation may be a vector in one place, and a quaternion in another. This amalgamation of the vectorial and quaternionic functions is very puzzling. You never know how things will turn out." †

"However, things changed as time went on, and after a period during which the diffusion of pure vectorial analysis made much progress, in spite of the disparagement of the Edinburgh school of scorners (one of whom said some of my work was 'a disgrace to the Royal Society,' to my great delight), it was most gratifying to find that Prof. Tait softened in his harsh judgments, and came to recognise the existence of rich fields of pure vector analysis, and to tolerate the workers therein. Besides those impertinent tamperers, Tait had to stick up for quaternionics against Cayley, for quite different reasons. There was danger of a triangular duel, or perhaps quadrangular, at

* *Ibid.* p. 301. † *Ibid.* pp. 302-3.

one time, but I would not engage in it for one. I appeased Tait considerably (during a little correspondence we had) by disclaiming any idea of discovering a new system. I professedly derived my system from Hamilton and Tait by elimination and simplification, but all the same claimed to have diffused a working knowledge of vectors, and to have devised a thoroughly practical system." *

Early in the present century, when the utility of Prof. Gibbs's system had been proved by twenty years' experience, he consented to its publication in a more extended form. Not having the leisure to undertake this work himself, he entrusted it to one of his former pupils, Dr. **Edwin Bidwell Wilson**, then instructor in Mathematics at Yale, now Professor of Mathematics in the Massachusetts Institute of Technology. Dr. Wilson was allowed free scope in his presentation of the subject.† He followed in the main the notation and methods of Gibbs, but adopted Heaviside's suggestion of Clarendon symbols for the representation of vectors. The success of his undertaking is well known. Professor Wilson has also contributed to the vector analysis of four dimensions arising in connection with the theory of Relativity. ‡

The present century has also witnessed the appearance of an Italian school of vector analysts, represented by Prof. **R. Marcolongo**, of the University of Naples, and Prof. **C. Burali-Forti**, of the Military Academy of Turin. Their vector algebra is substantially the same as that of other schools, with an independent notation for products of vectors; but both in this and in the differentiation of vectors the authors have been largely influenced by the geometric forms of Grassmann and the Barycentric Calculus of Möbius. For the linear vector function they have developed the properties of the *homographie* in place of the dyadic, being in this matter influenced chiefly by the work of Hamilton. The simpler portions of their system will be found in their *Eléments de Calcul Vectoriel* (Paris, 1910);

* *Electromagnetic Theory*, vol. 3, p. 137 (1912).

† *Vector Analysis* (1901). Yale University Press, 2nd ed. (1909).

‡ Wilson and Lewis, *Proc. Amer. Acad. of Arts and Sciences*, vol. 48 (1912). pp. 391-507

but for a full account the reader is referred to their larger work, *Analyse Vectorielle Générale* (Paris, 1912).

It is apparent then that the processes of current vector analysis have sprung from the work of Hamilton and Grassmann. The order of development of the subject has been the opposite of what one might have expected. "Suppose a sufficiently competent mathematician desired to find out from the Cartesian mathematics what vector algebra was like, and its laws. He could do so by careful inspection and comparison of the Cartesian formulae. He would find certain combinations of symbols and quantities occurring again and again, usually in systems of threes. He might introduce tentatively an abbreviated notation for these combinations. After a little practice he would perceive the laws according to which these combinations arose and how they operated. Finally, he would come to a very compact system in which vectors themselves and certain simple functions of vectors appeared, and would be delighted to find that the rules for the multiplication and general manipulation of these vectors were, considering the complexity of the Cartesian mathematics out of which he had discovered them, of an almost incredible simplicity. But there would be no sign of a quaternion in his result, for one thing; and, for another, there would be no metaphysics or abstruse reasoning required to establish the rules of manipulation of his vectors." * This is the manner in which one would expect Vector Analysis to have originated. But it did not; and its parentage has in many quarters counted against it. But this prejudice is rapidly disappearing, and the simple vector methods are becoming more and more popular. In advanced three-dimensional work in nearly every branch of mathematical physics, writers are finding it almost indispensable.

The lack of uniformity in the notation for products of vectors was more or less inevitable, but may yet be overcome. In America the dot and cross of Prof. Gibbs are employed almost without exception. In Germany the bracket notation is the general rule. The practice in England is by no means uniform. The influence of Lorentz's work has led some writers in the Electromagnetic Theory to follow his example in the use of

* Heaviside, *loc. cit.* vol. 1, p. 136.

brackets; while others adhere to the American usage. The author of these pages considers Gibbs's notation easier and much more elastic, especially for the treatment of linear vector functions. Gibbs's work on dyadics is the most original and important part of his theory, and will be found of great service in our second volume.

TABLE OF NOTATIONS *

	Vector	Scalar product.	Vector product.	Dyad.	Gradient.	Divergence.	Curl.
Gibbs, Wilson	$\mathbf{a}$	$\mathbf{a}\cdot\mathbf{b}$	$\mathbf{a}\times\mathbf{b}$	$\mathbf{ab}$	∇	$\nabla\cdot = \text{div}$	$\text{curl} = \nabla\times$
Heaviside - -	$\mathbf{a}$	$\mathbf{ab}$	$V\mathbf{ab}$	$\mathbf{a}.\mathbf{b}$	∇	div	curl
Abraham - -	$\mathfrak{A}$	$\mathfrak{AB}$	$[\mathfrak{AB}]$		∇	div	curl
Ignatowsky -	$\mathfrak{A}$	$\mathfrak{AB}$	$[\mathfrak{AB}]$	$\mathfrak{A};\mathfrak{B}$	∇	div	rot
Lorentz - -	$\mathbf{A}$	$(\mathbf{A}.\mathbf{B})$	$[\mathbf{A}.\mathbf{B}]$		grad	div	rot
Burali - Forti and Marcolongo	$\mathbf{a}$	$\mathbf{a}\times\mathbf{b}$	$\mathbf{a}\wedge\mathbf{b}$		grad	div	rot

SHORT COURSES.

General Short Course.

Beginners, and those who do not desire to go into the subject as fully as we have done in this book, are recommended to take the following short course:

Arts. 1-17, 19, 23, 24-31, 39-41, 42-46, 49-51, 55-62, 65-77, 82-85, 87-91, 97-100, 104.

Short Course for Mechanics.

Students who are interested more in the mechanical than in the geometrical applications are recommended to take the following course:

Arts. 1-17, 23, 24-29, 39-41, 42-46, 49-51, 54, 55-64, 65-79, 82-93, 97-101, 104.

Note.—In the text, Arts. marked with an asterisk * are intended only for the more advanced students.

* A similar table appears in the Introduction (p. 12) to *Le Calcul Vectoriel* by Guiot (Paris, 1912).

CHAPTER I.

ADDITION AND SUBTRACTION OF VECTORS. CENTROIDS.

Definitions.

1. A **scalar quantity**, or briefly a **scalar**, has magnitude, but is not related to any definite direction in space. Examples of such are mass, volume, density, temperature, work, quantity of heat, electric charge and potential. To specify a scalar we need a *unit* quantity of the same type, and the ratio (m) which the given quantity bears to this unit, so that it may be expressed as m times the unit. The number m is called the *measure* of the quantity in terms of the chosen unit. It is the measures d, m, V, v, E, etc., of density, mass, volume, speed and energy respectively, that enter into the equations of physics and mechanics; and it cannot be too strongly emphasized that these symbols denote only numbers, as in ordinary algebra.

A **vector quantity**, or briefly a **vector**, has magnitude and is related to a definite direction in space; while two vector quantities of the same kind are compounded according to the triangle law of addition stated below. Displacement, velocity, acceleration, momentum, force, electric and magnetic intensities are examples of vector quantities. To specify a vector we need not only a unit quantity of the same kind considered apart from direction, and a number which is the measure of the original quantity in terms of this unit, but also a statement of its direction.

2. Though ordinary algebra is adequate for the analysis of both scalars and vectors, when applied to the latter it is often very cumbrous, necessitating the manipulation of two or three equations instead of one, and the decomposition of the vector

quantities to meet the limitations of algebraic analysis. Hence the desirability of an analysis which will bear to vectors the same relation that algebra bears to scalars. In the latter the elements of our equations are always numbers, denoted by various symbols. In vector analysis we require, as well as these, elements involving both number and direction—directed numbers so to speak. For this purpose we choose what is perhaps the simplest type of vector quantity, viz. that whose magnitude is a length. A vector of this type is determined by two points O, P such that the magnitude of the vector is the length of the straight line OP, and its direction is that from O to P. This vector is usually denoted by $\overrightarrow{OP}$. For the purposes of analysis, after we have settled the units of the various types of vector quantities, any such quantity can be specified by a length-vector of the type $\overrightarrow{OP}$, provided their directions are the same, and the measure of the length of OP is also the measure of the vector quantity considered in terms of its appropriate unit. Briefly then, $\overrightarrow{OP}$ can be used to specify the vector quantity in measure and direction, and in this way answers the purpose of a directed number.

Having then decided to use length-vectors for our directed number elements, we shall find it convenient to abbreviate the name and call them simply *vectors*. To prevent confusion *we shall henceforth confine the substantive* **vector** *to length-vectors of this type. All others will be spoken of as* **vector quantities.** The practice of most writers seems to be in harmony with this, though in the majority of cases the restricted use of the term *vector* is only tacitly adopted.* We allow the wider meaning, but here adopt the narrower for the sake of brevity, and to avoid all possibility of misunderstanding.

3. The **module** of a vector is the positive number which is the measure of its length. A *unit vector* is one whose module is unity. We shall denote vectors by **Clarendon** letters,† and

* The term *vector* was invented by Hamilton; and the meaning he assigned to it agrees with the one we have just adopted. Cf. *Lectures on Quaternions*, Lecture 1, p. 15.

† For purposes of writing, Greek letters and script capitals will be found convenient to denote vectors.

their modules by the corresponding *italics*. Thus the vectors $\overrightarrow{PQ}$, $\overrightarrow{QR}$, $\overrightarrow{RS}$ may be denoted by **a**, **b**, **c** respectively, and their modules * by a, b, c. Unit vectors in these directions will be denoted by $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$, $\hat{\mathbf{c}}$ respectively.

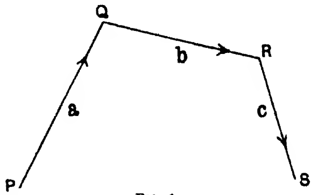

FIG. 1.

Two vectors **a**, **b** are defined to be *equal* if they have the same direction and equal lengths; and this is denoted symbolically by $\mathbf{a}=\mathbf{b}$, which is therefore equivalent to the two relations $\hat{\mathbf{a}}=\hat{\mathbf{b}}$ and $a=b$. A *zero vector*, or *null vector*, is one whose module is zero. All zero vectors are to be regarded as equal, irrespective of direction. *Like vectors* are vectors with the same direction. The vector which has the same module as **a** but the opposite direction is defined as the *negative* of **a**, and is denoted by $-\mathbf{a}$.

Thus, while the value of a vector depends on its length and direction, it is independent of position, the vector not being localised in any definite line. A single vector cannot therefore completely represent the effect of a localised vector quantity, such as a force acting on a rigid body. This effect depends on the line of action of the force; and it will be shown later that two vectors are necessary for its specification.

When vectors **F**, **v** are used to specify vector quantities such as force and velocity, they will always be length-vectors representing them in measure and direction. All the vectors entering into our equations will be of the same kind. For instance, in the equation $\mathbf{F}=m\mathbf{a}$, used for the mathematical expression of Newton's second law of motion, m is the measure of the mass of the particle, and **F**, **a** vectors representing respectively the force acting on the particle and the consequent acceleration.

* The notation mod **a** or $|\mathbf{a}|$ is also used for the module of **a**.

Addition and Subtraction of Vectors.

4. The manner in which the vector quantities of mechanics and physics are compounded is expressed by the **triangle law of addition**, which may be stated as follows:

If three points O, P, R are chosen so that $\overrightarrow{OP} = \mathbf{a}$ *and* $\overrightarrow{PR} = \mathbf{b}$, *then the vector* $\overrightarrow{OR}$ *is called the* (*vector*) *sum or resultant of* **a** *and* **b**.

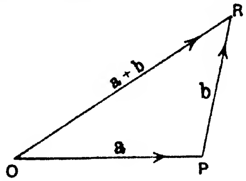

Fig. 2.

Denoting this resultant by **c**, we write

$$\mathbf{c} = \mathbf{a} + \mathbf{b},$$

borrowing the sign + from algebra, and using the term vector *addition* for the process by which the resultant **c** is obtained from the *components* **a** and **b**. The above definition is not an arbitrary mathematical assumption. It is an expression of the way in which the vector quantities of physics and mechanics are compounded. We see also that the sum of two vectors $\mathbf{a} = \overrightarrow{OP}$ and $\mathbf{b} = \overrightarrow{OQ}$ is the vector $\overrightarrow{OR}$ determined by the diagonal of the parallelogram of which OP and OQ are sides. For $\overrightarrow{PR} = \overrightarrow{OQ} = \mathbf{b}$, so that

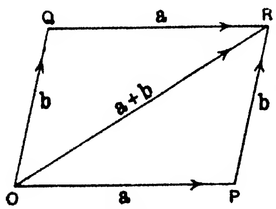

Fig. 3.

$$\mathbf{a} + \mathbf{b} = \overrightarrow{OP} + \overrightarrow{PR} = \overrightarrow{OR}.$$

Thus the triangle law of addition is identical with the parallelogram law involved in the so-called "parallelogram of forces."

Further, since $\overrightarrow{QR} = \overrightarrow{OP} = \mathbf{a}$ it follows that

$$\mathbf{b} + \mathbf{a} = \overrightarrow{OQ} + \overrightarrow{QR} = \overrightarrow{OR},$$

showing that $\qquad \mathbf{b} + \mathbf{a} = \mathbf{a} + \mathbf{b} = \mathbf{r}$ (say).

Again, we may add to this another vector $\mathbf{c} = \overrightarrow{RS}$, obtaining the result

$$\overrightarrow{OS} = \mathbf{r} + \mathbf{c} = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$$
$$= \mathbf{c} + (\mathbf{a} + \mathbf{b}).$$

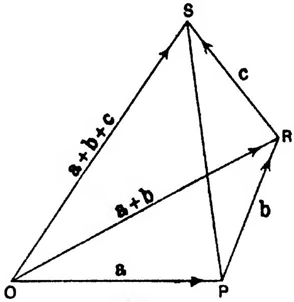

Fig. 4.

But a glance at the figure shows that this vector is also

$$\overrightarrow{OS} = \overrightarrow{OP} + \overrightarrow{PS} = \mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{b} + \mathbf{c}) + \mathbf{a},$$

and the argument obviously holds for any number of vectors. Hence the

Theorem. *The commutative and associative laws hold for the addition of any number of vectors. The sum is independent of the order and the grouping of the terms.*

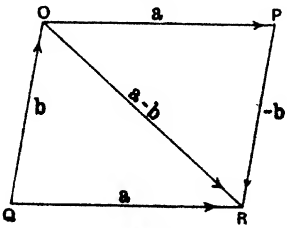

Fig. 5.

We have already stated that $-\mathbf{b}$ is to be understood as the vector which has the same length as $\mathbf{b}$, but the opposite direction. The subtraction of $\mathbf{b}$ from $\mathbf{a}$ is to be understood as the addition of $-\mathbf{b}$ to $\mathbf{a}$. We denote this by

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}),$$

borrowing the $-$ sign from algebra. Thus *to subtract the vector* **b** *from* **a**, *reverse the direction of* **b** *and add.* If, in the figure,

$$\overrightarrow{OP} = \overrightarrow{QR} = \mathbf{a} \quad \text{and} \quad \overrightarrow{QO} = \overrightarrow{RP} = \mathbf{b},$$

then
$$\mathbf{a} - \mathbf{b} = \overrightarrow{OP} + \overrightarrow{PR} = \overrightarrow{OR}.$$

5. Multiplication by a number. *If m is any positive real number,* $m\mathbf{a}$ means the vector in the same direction as* **a**, *but of m times its length.* This may be regarded as the result of multiplying **a** by m; and similarly $\frac{1}{m}\mathbf{a}$ is the result of dividing **a** by m.

In agreement with the preceding Art., $-m\mathbf{a}$ is then the vector in the opposite direction to **a**, and of m times its length. Thus, to multiply a vector by a negative real number $-m$, reverse its direction and multiply by m.

From the above it is clear that, if **a** and **b** are *like* vectors, either may be expressed as a multiple of the other. Thus $\mathbf{b} = \frac{b}{a}\mathbf{a}$, the number $\frac{b}{a}$ being the ratio of the length of **b** to that of **a**. In particular if $\hat{\mathbf{a}}$ is the unit vector in the direction of **a** then

$$\mathbf{a} = a\hat{\mathbf{a}} \quad \text{and} \quad m\mathbf{a} = m(a\hat{\mathbf{a}}) = (ma)\hat{\mathbf{a}}.$$

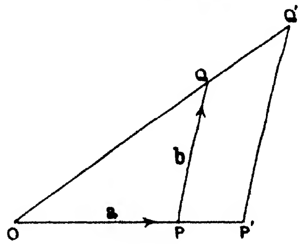

FIG. 6.

The general laws of association and distribution for scalar multipliers hold as in ordinary algebra. If m and n are any real numbers, positive or negative, it follows from the above argument that

$$m(n\mathbf{a}) = (mn)\mathbf{a} = n(m\mathbf{a}),$$

and also that

$$(m+n)\mathbf{a} = m\mathbf{a} + n\mathbf{a}.$$

* Imaginary and complex numbers are excluded from our discussion.

Lastly, the formula $\quad m(\mathbf{a}+\mathbf{b})=m\mathbf{a}+m\mathbf{b}$

is easily proved geometrically. For if $\overrightarrow{OP}=\mathbf{a}$ and $\overrightarrow{PQ}=\mathbf{b}$, then $\overrightarrow{OQ}=\mathbf{a}+\mathbf{b}$. But if P', Q' are points in OP and OQ respectively so that $OP':OP=OQ':OQ=m$, then $P'Q'$ is parallel to PQ and m times it in length. Thus $\overrightarrow{P'Q'}=m\mathbf{b}$, showing that

$$m(\mathbf{a}+\mathbf{b})=\overrightarrow{OQ'}=\overrightarrow{OP'}+\overrightarrow{P'Q'}=m\mathbf{a}+m\mathbf{b}.$$

Components of a Vector.

6. Three or more vectors are said to be *coplanar* when a plane can be drawn parallel to all of them; otherwise they are *non-coplanar*.

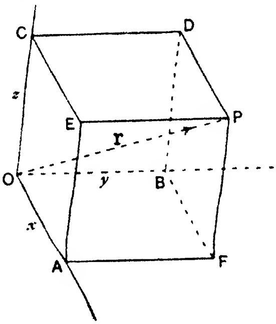

Fig. 7

Any vector $\mathbf{r}$ can be expressed as the sum of three others, parallel to any three non-coplanar vectors. Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be *unit* vectors in the three given non-coplanar directions. With any point O as origin take $\overrightarrow{OP}=\mathbf{r}$, and on OP as diagonal construct a parallelepiped with edges OA, OB, OC parallel to $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ respectively. Then if x, y, z are the measures of the lengths of its edges, $\mathbf{r}$ is expressible as the sum

$$\begin{aligned}\mathbf{r}&=\overrightarrow{OA}+\overrightarrow{AF}+\overrightarrow{FP}=\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}\\&=x\mathbf{a}+y\mathbf{b}+z\mathbf{c}.\end{aligned}$$

Thus $\mathbf{r}$ is the *resultant* of the three vectors $x\mathbf{a}$, $y\mathbf{b}$, $z\mathbf{c}$, which are called the *components* of $\mathbf{r}$ in the given directions.* This resolution of $\mathbf{r}$ is unique, because only one parallelepiped can be constructed on OP as diagonal with edges parallel to the given directions. Hence, if two vectors are equal the components

* The numbers x, y, z may be either positive or negative. For instance, x will be positive if the component of $\overrightarrow{OA}$ has the same direction as $\mathbf{a}$; negative if the opposite direction.

of the one are equal to those of the other, each to each. Conversely, if two vectors have equal components they must be equal.

Given several vectors $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$, ... they may each be resolved into components in the given directions, and expressed in the form

$$\mathbf{r}_1 = x_1\mathbf{a} + y_1\mathbf{b} + z_1\mathbf{c},$$
$$\mathbf{r}_2 = x_2\mathbf{a} + y_2\mathbf{b} + z_2\mathbf{c},$$
$$\cdots\cdots\cdots$$

Their sum is then

$$\begin{aligned}(x_1\mathbf{a} + y_1\mathbf{b} + z_1\mathbf{c}) + (x_2\mathbf{a} + y_2\mathbf{b} + z_2\mathbf{c}) + \ldots \\ = (x_1 + x_2 + x_3 + \ldots)\mathbf{a} + (y_1 + y_2 + \ldots)\mathbf{b} \\ + (z_1 + z_2 + \ldots)\mathbf{c},\end{aligned}$$

showing that vectors may be compounded by adding their like components.

7. The unit vectors i, j, k. The most important case of resolution of vectors is that in which the three directions are

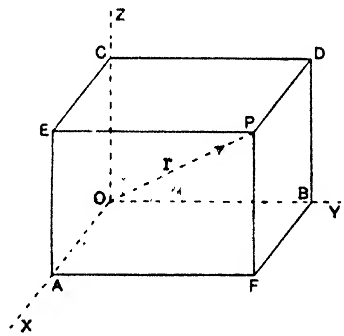

FIG. 8.

mutually perpendicular. The right-handed system of directions OX, OY, OZ represented in the figure is found most convenient; OY and OZ are in the plane of the paper, and OX perpendicular to it pointing toward the reader. To an observer at the origin O, right-handed rotations about the axes OX, OY, OZ are from Y to Z, Z to X and X to Y respectively. The unit vectors parallel to these axes are denoted by $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$; and if x, y, z are the lengths of OA, OB, OC respectively measured in these directions, the vector $\overrightarrow{OP}$ is

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

If α, β, γ are the angles which $\overrightarrow{OP}$ makes with the axes, $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are called the *direction cosines* of the line OP; and clearly

$$x=r\cos\alpha,\quad y=r\cos\beta,\quad z=r\cos\gamma,$$

so that

$$\hat{\mathbf{r}}=\frac{1}{r}\mathbf{r}=(\cos\alpha)\mathbf{i}+(\cos\beta)\mathbf{j}+(\cos\gamma)\mathbf{k}.$$

Thus the coefficients of $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ in the rectangular resolution of a unit vector are the direction cosines of that vector relative to the rectangular axes. Rectangular components are generally termed *resolutes* or **resolved** *parts*.

The numbers x, y, z are called the *coordinates* of the point P relative to the axes OX, OY, OZ. It is obvious from the figure that

$$OP^2=OA^2+AF^2+FP^2,$$

that is

$$r^2=x^2+y^2+z^2,$$

giving the distance of P from the origin in terms of its coordinates.

In Art. 6. where the axes are oblique, x, y, z are still called the coordinates relative to those axes; but the expression for r in terms of these involves also the mutual inclinations of the axes. (Cf. Exercise (4), Art. 26.)

Cartesian analysis deals with vectors and vector quantities by resolving them into rectangular components. In vector analysis, as far as possible, we treat the quantities without resolution.

Centroids.

Definition. *When a vector $\overrightarrow{OP}$ is used to specify the position of a point P relative to another point O, it is called the* **position vector** *of P for the origin O.*

8. *To find the point which divides the join of two points in a given ratio.*

Let A, B be the two points and $\mathbf{a}$, $\mathbf{b}$ their position vectors relative to an origin O. Then $\overrightarrow{AB}=\mathbf{b}-\mathbf{a}$; and if R is the point dividing AB in the ratio $m:n$ it follows that

$$\overrightarrow{AR}=\frac{m}{m+n}(\mathbf{b}-\mathbf{a}).$$

The position vector of R is therefore

$$\mathbf{r}=\overrightarrow{OR}=\overrightarrow{OA}+\overrightarrow{AR}$$

$$=\mathbf{a}+\frac{m}{m+n}(\mathbf{b}-\mathbf{a})$$

$$=\frac{n\mathbf{a}+m\mathbf{b}}{m+n}.$$

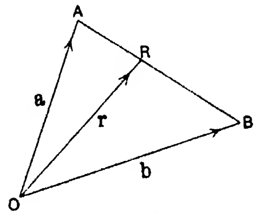

FIG. 9.

9. Definitions.* *Given n points whose position vectors relative to an origin O are* **a**, **b**, **c**, ... , *the point G whose position vector is*

$$\overrightarrow{OG}=\frac{1}{n}(\mathbf{a}+\mathbf{b}+\mathbf{c}+\ldots)$$

is called the **centroid** *or* **centre of mean position** *of the given points*

If $p, q, r, \ldots$ are n real numbers, the point G whose position vector is

$$\overrightarrow{OG}=\frac{p\mathbf{a}+q\mathbf{b}+r\mathbf{c}+\ldots}{p+q+r+\ldots}$$

is called the centroid of the given points with associated numbers† *$p, q, r, \ldots$ respectively.*

The centroid of two points A, B with associated numbers p, q divides the line AB in the ratio $q:p$. For in this case,

$$\overrightarrow{OG}=\frac{p\mathbf{a}+q\mathbf{b}}{p+q},$$

which proves the statement.

* Regarding this treatment of the theory of centroids see remarks in the Preface.

† The term *strength* may be found more convenient than "associated number."

Theorem. *The centroid is independent of the origin of vectors.*

Let O' be a point whose position vector relative to O is $\mathbf{l}$. If O' is taken as origin, the position vectors of the points A, B, C, ... are $\mathbf{a}-\mathbf{l}$, $\mathbf{b}-\mathbf{l}$, $\mathbf{c}-\mathbf{l}$, ... ; and the centroid is a point G' such that

$$\overrightarrow{O'G'} = \frac{p(\mathbf{a}-\mathbf{l})+q(\mathbf{b}-\mathbf{l})+\dots}{p+q+r+\dots}$$

$$= \frac{p\mathbf{a}+q\mathbf{b}+r\mathbf{c}+\dots}{p+q+r} - \mathbf{l}$$

$$= \overrightarrow{OG} - \mathbf{l} = \overrightarrow{O'G}.$$

Hence the points G, G' coincide, and the position found for the centroid is independent of the origin of vectors.

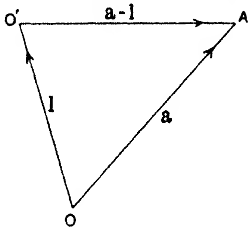

FIG. 10.

Theorem. *If H is the centroid of a system of points A, B, C, ... with associated numbers p, q, r, ... , and H' that of a second system of points A', B', C', ... with associated numbers p', q', r', ... , then the centroid of all the points is the centroid of the two points H and H' with associated numbers*

$$(p+q+r+\dots) \quad \textit{and} \quad (p'+q'+r'+\dots).$$

For $$\overrightarrow{OH} = \frac{p\mathbf{a}+q\mathbf{b}+r\mathbf{c}+\dots}{p+q+r+\dots} = \frac{\Sigma p\mathbf{a}}{\Sigma p},$$

and similarly $$\overrightarrow{OH'} = \frac{p'\mathbf{a}'+q'\mathbf{b}'+r'\mathbf{c}'+\dots}{p'+q'+r'+\dots} = \frac{\Sigma p'\mathbf{a}'}{\Sigma p'}.$$

Hence the centroid of H and H' with associated numbers Σp and $\Sigma p'$ respectively is a point G such that

$$\overrightarrow{OG} = \frac{(\Sigma p)\overrightarrow{OH}+(\Sigma p')\overrightarrow{OH'}}{\Sigma p+\Sigma p'} = \frac{\Sigma p\mathbf{a}+\Sigma p'\mathbf{a}'}{\Sigma p+\Sigma p'}.$$

Hence G is also the centroid of the combined system of points.

The theorem has been stated for only two sub-systems of points with centroids H and H'. But the same argument applies, and the theorem is true, for any number of sub-systems. In calculating the centroid of the combined system of points, each sub-system may be replaced by a single point (its centroid) with associated number Σp for that sub-system.

10. Centroid of Area. Suppose the surface of any figure (not necessarily plane) to be divided into a large number n of small elements. Consider one point in each element, and with these n points associate numbers proportional to the areas of the elements. Such a system of points with associated numbers has a centroid G. Now let the number n increase indefinitely, in such a way that each element of the surface converges to a point. Then the limiting position of G is called the centroid of area of the figure.

Centroid of Volume. Suppose any solid figure to be divided into a large number n of small elements. Consider one point in each element, and with these n points associate numbers proportional to the volumes of the elements. Such a system of points has a centroid G; and the limiting position of G as n tends to infinity and each element converges to a point, is called the centroid of volume of the figure.

The centroids of area and volume of the simpler figures of plane and solid geometry are easily found by considerations of symmetry, without the need of introducing vectors. Students of geometry and mechanics become familiar with these results at an early stage, and we shall not here enter upon their proofs. The determination of centroids by integration of vectors will be referred to in Art. 62.

11. Definition. *The* **centroid of mass**, *or* **centre of mass**, *of a set of particles of masses* $m_1, m_2, m_3, \ldots$ *situated at the points** $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \ldots$ *respectively is the centroid of these points with associated numbers* $m_1, m_2, m_3, \ldots$.

The centre of mass (c.m.) of the system of particles is therefore the point

$$\bar{\mathbf{r}} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + \ldots}{m_1 + m_2 + \ldots} = \frac{\Sigma m\mathbf{r}}{\Sigma m}.$$

* When the origin of vectors is understood, the point whose position vector is $\mathbf{r}$ may be conveniently referred to as the point $\mathbf{r}$.

It will be shown in Art. 99 that this is the point through which passes the line of action of the resultant of any system of parallel forces acting on the particles, the forces being proportional to the masses of the particles. It will follow then that the c.m. as defined above is identical with the centre of gravity of the system of particles. Being a centroid, the c.m. is independent of the origin of vectors. Also, if the system of particles be divided into n sub-systems, the c.m. of the whole system is the c.m. of n particles, one at the c.m. of each sub-system and of mass equal to the total mass of that sub-system. This follows from the second theorem of Art. 9

From the above formula for the c.m. of a system of particles we may easily deduce the usual *scalar equations*. Take a set of axes through the origin parallel to the unit vectors **a**, **b**, **c**. Let x, y, z be the coordinates of a particle of mass m, and $\bar{x}$, $\bar{y}$, $\bar{z}$ those of the c.m. Then the position vectors of these points are

$$\mathbf{r} = x\mathbf{a} + y\mathbf{b} + z\mathbf{c}$$

and

$$\bar{\mathbf{r}} = \bar{x}\mathbf{a} + \bar{y}\mathbf{b} + \bar{z}\mathbf{c}$$

The formula for the c.m. is therefore

$$\bar{x}\mathbf{a} + \bar{y}\mathbf{b} + \bar{z}\mathbf{c} = \frac{\Sigma m(x\mathbf{a} + y\mathbf{b} + z\mathbf{c})}{\Sigma m}$$

The equal vectors represented by the two members of this equation must have equal components. Hence, equating the coefficients of like unit vectors, we have

$$\bar{x} = \frac{\Sigma mx}{\Sigma m}, \quad \bar{y} = \frac{\Sigma my}{\Sigma m}, \quad \bar{z} = \frac{\Sigma mz}{\Sigma m},$$

and these formulae are true whether the axes are rectangular or oblique.

The c.m. of a **continuous distribution of matter**, whether a surface or a volume distribution, is defined as follows. Let the distribution be divided into a large number n of small elements. Take n points, one situated in each element, and with these associate numbers proportional to the masses of the elements. This system of n points with associated numbers has a centroid G. Now let n tend to infinity in such a way that each of the elements of mass converges to a particle. The limiting position of G is called the centre of mass of the continuous distribution.

When the density of the distribution is uniform, the c.m. coincides with the centroid of the geometrical figure occupied by the distribution.

Elementary Physical Applications.

12. The **displacement** of a point is a vector quantity, a displacement from A to B being specified in magnitude and direction by the vector $\overrightarrow{AB}$. And two successive displacements are compounded according to the vector law of addition; for a displacement from A to B followed by another from B to C brings the moving point to the same position as a single displacement from A to C.

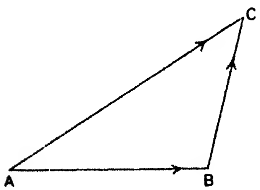

FIG. 11.

The **relative position** of one point Q to another P, both of which may be in motion, is a vector quantity specified by the vector $\overrightarrow{PQ}$. And clearly the relative position of Q to a third point O is the vector sum of the relative positions of Q to P and of P to O; for $\overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PQ}$.

The **relative displacement** of Q to P during any interval is the change in relative position during that interval. If then P, Q are the positions of the points at the beginning, and P', Q' at the end of the interval, the change in relative position is determined by the vector difference $\overrightarrow{P'Q'} - \overrightarrow{PQ}$. This vector specifies the relative displacement during the interval.

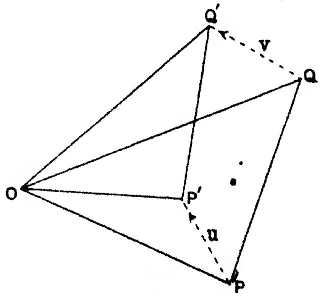

FIG. 12.

The **relative velocity** of Q with respect to P is the rate of change of its position relative to P, and therefore, if *uniform*, is determined

by the relative displacement in one second. This relative velocity is the vector difference of the velocities of Q and P relative to another point O (which may be regarded as fixed). For, supposing the velocities uniform, let the points be displaced in one second from P, Q to P', Q'. The vector specifying their relative velocity is that which represents the change in their relative position in unit time, *i.e.* $\overrightarrow{P'Q'} - \overrightarrow{PQ}$, which may also be expressed

$$\overrightarrow{OQ'} - \overrightarrow{OP'} - (\overrightarrow{OQ} - \overrightarrow{OP}) = (\overrightarrow{OQ'} - \overrightarrow{OQ}) - (\overrightarrow{OP'} - \overrightarrow{OP})$$
$$= \overrightarrow{QQ'} - \overrightarrow{PP'} = \mathbf{v} - \mathbf{u},$$

where $\mathbf{v}$, $\mathbf{u}$ are the velocity vectors for Q and P respectively relative to O. Hence the velocity of Q relative to P is the vector difference of the velocities of Q and P relative to O.

The case of *variable velocities* and relative acceleration will be considered in Chapter VI., after differentiation of vectors has been dealt with.

13. Concurrent forces. A force has magnitude and direction, and may be represented in these respects by a vector. But a force has also a definite line of action, and its effect upon a body is altered if this line of action is changed, even though the direction may not vary. We shall for the present confine our attention to forces whose lines of action are concurrent, as for instance when all the forces act on a single particle. Now, experiment shows that the joint action of two concurrent forces has the same dynamical effect as that of a single force which is equal to their vector sum, and acts through their point of concurrence. And if there are several forces acting on a body, represented by the vectors $\mathbf{F}_1$, $\mathbf{F}_2$, . . . respectively, and with lines of action concurrent at a point P, the single force represented by

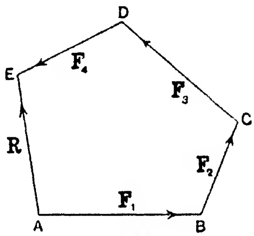

FIG. 13.

$$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \ldots = \Sigma\mathbf{F},$$

and acting through the same point P, is dynamically equivalent to the system of forces, and is called their *resultant*.

The vector $\mathbf{R}$ is determined by the *vector polygon*, that is a polygon the lengths and directions of whose sides are those of the vectors $\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3, \ldots$. This polygon will not in general be closed, nor will it be plane unless the forces are coplanar. If $\overrightarrow{AB}$ is the first vector and $\overrightarrow{DE}$ the last, then $\overrightarrow{AE}$ is the resultant

$$\mathbf{R} \equiv \Sigma\mathbf{F}.$$

When the vector sum of all the forces is zero, they are together equivalent to zero force, and are said to be in equilibrium; or, the particle or body on which they act is said to be in equilibrium under the action of the forces. In this case the vector polygon is closed. And since the resultant vanishes, the sums of the components of the several forces in any three non-coplanar directions must vanish separately. (Cf. Art. 6.) Conversely, if the sum of the components of the forces vanishes for each of three non-coplanar directions, the components of the resultant are zero, and the resultant vanishes. This then is the necessary and sufficient condition for equilibrium of the forces.

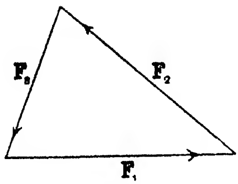

FIG. 14.

If *three forces* acting at a point are in equilibrium, the closed vector polygon is a triangle. The vectors $\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3$ are then coplanar, and the length of each is proportional to the sine of the angle between the other two. This is **Lami's Theorem**, viz.: *If three concurrent forces are in equilibrium they are coplanar, and each is proportional to the sine of the angle between the other two.*

In the case of n concurrent forces, let $A_1, A_2, A_3, \ldots, A_n$ be the points whose position vectors are $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n$ relative to an origin O. Then the vector representing the resultant is

$$\mathbf{R} = \Sigma\mathbf{F} = \overrightarrow{OA_1} + \overrightarrow{OA_2} + \ldots + \overrightarrow{OA_n}$$
$$= n \cdot \overrightarrow{OG},$$

where G is the centroid of the points $A_1, A_2, \ldots, A_n$. The forces are in equilibrium if G coincides with O.

14. Examples.

(1) *A man travelling East at 8 miles an hour finds that the wind seems to blow directly from the North. On doubling his speed he finds that it appears to come from N.E. Find the velocity of the wind.*

Let $\mathbf{i}$, $\mathbf{j}$ represent velocities of 8 miles an hour toward E. and N. respectively. Then the original velocity of the man is $\mathbf{i}$. Let that of the wind be $x\mathbf{i}+y\mathbf{j}$. Then the velocity of the wind relative to the man is

$$(x\mathbf{i}+y\mathbf{j})-\mathbf{i}.$$

But this is from the N., and is therefore parallel to $-\mathbf{j}$. Hence $x=1$.

When the man doubles his speed the velocity of the wind relative to him is

$$(x\mathbf{i}+y\mathbf{j})-2\mathbf{i}.$$

But this is from N.E., and is therefore parallel to $-(\mathbf{i}+\mathbf{j})$. Hence

$$y=x-2=-1.$$

Thus the velocity of the wind is $\mathbf{i}-\mathbf{j}$, which is equivalent to $8\sqrt{2}$ miles an hour from N.W.

(2) *If two concurrent forces are represented by* $n\,.\,\overrightarrow{OA}$ *and* $m\,.\,\overrightarrow{OB}$ *respectively, their resultant is given by* $(m+n)\,\overrightarrow{OR}$, *where* R *divides* AB *so that*

$$n\,.\,AR=m\,.\,RB.$$

By reference to Fig. 9 it will be seen that

$$\overrightarrow{OA}=\overrightarrow{OR}+\overrightarrow{RA},$$

and

$$\overrightarrow{OB}=\overrightarrow{OR}+\overrightarrow{RB}.$$

Hence the resultant of the forces $n\,.\,\overrightarrow{OA}$ and $m\,.\,\overrightarrow{OB}$ is

$$(m+n)\overrightarrow{OR}+(n\,.\,\overrightarrow{RA}+m\,.\,\overrightarrow{RB}).$$

And the last part of this expression is zero because R divides AB in the ratio $m:n$.

(3) $P_1, P_2, \ldots P_n$ *are* n *points dividing the circumference of a circle into* n *equal parts. Find the resultant of forces represented by* $\overrightarrow{AP_1}, \overrightarrow{AP_2}, \ldots, \overrightarrow{AP_n}$ *where* A *is any point, not necessarily in the plane of the circle.*

By the preceding Art. the resultant is represented by $n\,.\,\overrightarrow{AG}$, where G is the centroid of the points $P_1, P_2, \ldots, P_n$. And by symmetry G coincides with the centre O of the circle.

(4) *A particle is acted on by a number of centres of force, some of which attract and some repel, the force in each case varying as the distance,*

and the intensities for different centres being different. Show that the resultant passes through a fixed point for all positions of the particle.

Let P be the position of the particle, and O_1, O_2, . . . those of the centres of force. The forces on the particle due to the different centres are represented by

$$\mu_1 . \overrightarrow{PO_1}, \quad \mu_2 . \overrightarrow{PO_2}, \quad \text{etc.},$$

where $\mu_1, \mu_2, \ldots$ are constants, positive for the centres that attract and negative for those that repel. The resultant force on the particle is

$$\mu_1 . \overrightarrow{PO_1} + \mu_2 . \overrightarrow{PO_2} + \ldots = (\mu_1 + \mu_2 + \ldots)\overrightarrow{PG},$$

where G is the centroid of the points $O_1, O_2, O_3, \ldots$ with associated numbers $\mu_1, \mu_2, \ldots$ respectively. And this is a fixed point independent of the position of P.

EXERCISES ON CHAPTER I.

1. Find the sum of the vectors $3\mathbf{i}+7\mathbf{j}-4\mathbf{k}$, $\mathbf{i}-5\mathbf{j}-8\mathbf{k}$ and $6\mathbf{i}-2\mathbf{j}+12\mathbf{k}$. Also calculate the module and direction cosines of each.

2. If the position vectors of P and Q are $\mathbf{i}+3\mathbf{j}-7\mathbf{k}$ and $5\mathbf{i}-2\mathbf{j}+4\mathbf{k}$ respectively, find $\overrightarrow{PQ}$ and determine its direction cosines.

3. If the vertices of a triangle are the points

$$a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}, \quad b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k} \quad \text{and} \quad c_1\mathbf{i}+c_2\mathbf{j}+c_3\mathbf{k},$$

what are the vectors determined by its sides? Find the lengths of these vectors.

4. The position vectors of the four points A, B, C, D are

$$\mathbf{a},\ \mathbf{b},\ 2\mathbf{a}+3\mathbf{b} \text{ and } \mathbf{a}-2\mathbf{b} \text{ respectively.}$$

Express $\overrightarrow{AC}$, $\overrightarrow{DB}$, $\overrightarrow{BC}$ and $\overrightarrow{CA}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

5. If $\mathbf{a}$, $\mathbf{b}$ are the vectors determined by two adjacent sides of a regular hexagon, what are the vectors determined by the other sides taken in order?

6. A point describes a circle uniformly in the $\mathbf{i}$, $\mathbf{j}$ plane taking 12 seconds to complete one revolution. If its initial position vector relative to the centre is $\mathbf{i}$, and the rotation is from $\mathbf{i}$ to $\mathbf{j}$, find the position vectors at the end of 1, 3, 5, 7 seconds; also at the end of $1\frac{1}{2}$ and $4\frac{1}{2}$ seconds.

7. In the previous exercise find the velocity vectors of the moving point at the end of $1\frac{1}{2}$, 3 and 7 seconds.

8. The velocity of a boat relative to the water is represented by $3\mathbf{i}+4\mathbf{j}$, and that of the water relative to the earth by $\mathbf{i}-3\mathbf{j}$. What is the velocity of the boat relative to the earth if $\mathbf{i}$ and $\mathbf{j}$ represent velocities of one mile an hour E. and N. respectively ?

✓ 9. Two particles are moving with the same speed v ft./sec., one along a fixed diameter of a circle and the other round its circumference. Find the velocity of the first relative to the second when the radius to the latter makes an angle θ with the direction of motion of the former, θ *increasing*. [Assume that variable velocities obey the same law of composition as uniform velocities. Cf. Art. 65.]

✓ 10. Two particles, instantaneously at A and B respectively, 15 feet apart, are moving with uniform velocities, the former toward B at 5 ft./sec., and the latter perpendicular to AB at $3\frac{3}{4}$ ft./sec. Find their relative velocity, their shortest distance apart, and the instant when they are nearest.

11. Find the sum of the three vectors determined by the diagonals of three adjacent faces of a cube passing through the same corner, the vectors being directed from that corner.

12. A particle at the corner of a cube is acted on by forces 1, 2, 3 lb. wt. respectively along the diagonals of the faces of the cube which meet at the particle. Find their resultant.

13. Find the horizontal force and the force inclined at 60° to the vertical, whose resultant is a vertical force P lb. wt.

14. If the resultant of two forces is equal in magnitude to one of the components, and perpendicular to it in direction, find the other component.

✓ 15. Two forces act at the corner A of a quadrilateral $ABCD$, represented by $\overrightarrow{AB}$ and $\overrightarrow{AD}$; and two at C represented by $\overrightarrow{CB}$ and $\overrightarrow{CD}$. Show that their resultant is represented by $4\overrightarrow{PQ}$, where P, Q are the mid points of AC, BD respectively.

16. Find the c.m. of particles of masses 1, 2, 3, 4, 5, 6, 7, 8 grams respectively, placed at the corners of a unit cube, the first four at the corners A, B, C, D of one face, and the last four at their projections A', B', C', D' respectively on the opposite face.

17. Find the centroid of the $3n$ points $\mathbf{i}, 2\mathbf{i}, 3\mathbf{i}, \ldots, n\mathbf{i}$; $\mathbf{j}, 2\mathbf{j}, 3\mathbf{j}, \ldots, n\mathbf{j}$; $\mathbf{k}, 2\mathbf{k}, \ldots, n\mathbf{k}$.

18. Five forces act at one vertex A of a regular hexagon in the directions of the other vertices, and proportional to the distances of those vertices from A. Find their resultant.

19. If O is the circumcentre and O' the orthocentre of a triangle ABC, prove that the sum of the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ is $\overrightarrow{OO'}$; and that the sum of $\overrightarrow{O'A}$, $\overrightarrow{O'B}$, $\overrightarrow{O'C}$ is $2 . \overrightarrow{O'O}$.

20. In the previous exercise show that the sum of $\overrightarrow{AO'}$, $\overrightarrow{O'B}$, $\overrightarrow{O'C}$ is $\overrightarrow{AD}$, where AD is a diameter of the circumcircle.

21. D, E, F are the mid points of the sides of the triangle A, B, C. Show that, for any point O, the system of concurrent forces represented by $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ is equivalent to the system represented by $\overrightarrow{OD}$, $\overrightarrow{OE}$, $\overrightarrow{OF}$ acting at the same point.

22. If **a**, **b** are the position vectors of A, B respectively, find that of a point C in AB produced such that $AC=3AB$; and that of a point D in BA produced such that $BD=2BA$.

23. A, B, C are fixed points and P a variable point such that the resultant of forces at P represented by $\overrightarrow{PA}$ and $\overrightarrow{PB}$ always passes through C. Find the locus of P.

24. ABC is a triangle and P any point in BC. If $\overrightarrow{PQ}$ is the resultant of $\overrightarrow{AP}$, $\overrightarrow{PB}$, $\overrightarrow{PC}$, the locus of Q is a straight line parallel to BC.

25. Prove that the magnitude of the resultant of any number of forces P_1, P_2, P_3 ... is given by

$$R^2 = \Sigma P^2 + 2\Sigma P_r P_s \cos (P_r P_s).$$

26. Forces P, Q act at O and have a resultant R. If any transversal cuts their lines of action at A, B, C respectively, show that

$$\frac{P}{OA} + \frac{Q}{OB} = \frac{R}{OC}.$$

27. Between any four non-coplanar vectors there exists one, and only one, linear relation.

28. Particles of equal mass are placed at $(n-2)$ of the corners of a regular polygon of n sides. Find their c.m.

29. A line AB is bisected in P_1, P_1B in P_2, P_2B in P_3, and so on *ad infinitum*; and particles whose masses are m, $\frac{1}{2}m$, $\frac{1}{2^2}m$, etc., are placed at the points P_1, P_2, P_3, etc. Prove that the distance of their c.m. from B is equal to one-third of the distance from A to B.

CHAPTER II.

ELEMENTARY GEOMETRICAL ILLUSTRATIONS AND APPLICATIONS.

15. In the present chapter we shall consider the vector equation of a straight line, one form of vector equation for a plane, and the quantity called "vector area." In connection with the first of these, some examples are taken from elementary plane geometry. This, however, is merely for the purpose of illustrating the vector method, and not with the intention of recommending the use of vectors in elementary geometry. Vectors were designed essentially for three-dimensional calculations; and the author explicitly disowns any attempt to recommend their use in elementary two-dimensional problems, especially of geometry.

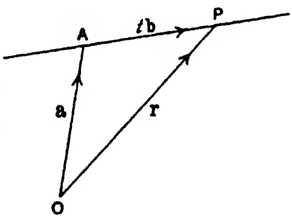

FIG. 15.

16. Vector equation of a straight line. *To find the vector equation of the straight line through a given point A parallel to a given vector* $\mathbf{b}$.

If P is a point on this straight line the vector $\overrightarrow{AP}$ is parallel to $\mathbf{b}$, and is therefore equal to $t\mathbf{b}$, where t is some real number positive for points on one side of A, and negative for points on the other, varying from point to point. Thus, if $\mathbf{a}$ is the position vector of A, that of P is

$$\mathbf{r} = \overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$$

$$= \mathbf{a} + t\mathbf{b}. \quad \text{.....................................(1)}$$

And since any point on the given straight line has a position vector represented by this equation for some value of t, we speak of (1) as the vector equation of the straight line. It is also clear that for all values of t the point $\mathbf{a}+t\mathbf{b}$ lies on the given line.

Cor. The vector equation of the straight line through the origin parallel to $\mathbf{b}$ is

$$\mathbf{r}=t\mathbf{b}. \qquad \text{.................................(2)}$$

[If (x, y, z) and (a_1, a_2, a_3) are the coordinates of P and A respectively referred to rectangular axes through the origin O, while $\mathbf{b}=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$, the above equation may be written

$$x\mathbf{i}+y\mathbf{j}+z\mathbf{k}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}+t(b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}).$$

Then, equating coefficients of like vectors on opposite sides of this equation, we deduce the relations

$$\frac{x-a_1}{b_1}=\frac{y-a_2}{b_2}=\frac{z-a_3}{b_3}=t,$$

which are the ordinary equations of coordinate geometry for the straight line through the point (a_1, a_2, a_3) with direction cosines proportional to (b_1, b_2, b_3).]

To find the vector equation of the straight line passing through the points A and B, whose position vectors are $\mathbf{a}$ and $\mathbf{b}$, we

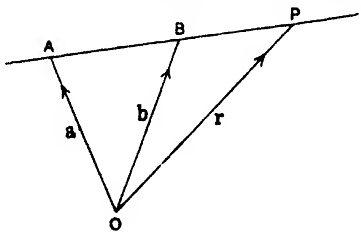

FIG. 16.

observe that $\overrightarrow{AB}=\mathbf{b}-\mathbf{a}$; so that the straight line is one through the point A parallel to $\mathbf{b}-\mathbf{a}$. Its vector equation is therefore

$$\mathbf{r}=\mathbf{a}+t(\mathbf{b}-\mathbf{a})$$

or

$$\mathbf{r}=(1-t)\mathbf{a}+t\mathbf{b}. \qquad \text{..........................(3)}$$

[If (a_1, a_2, a_3) and (b_1, b_2, b_3) are the coordinates of A and B, we find as above that the equation (3) is equivalent to the relations

$$\frac{x-a_1}{b_1-a_1}=\frac{y-a_2}{b_2-a_2}=\frac{z-a_3}{b_3-a_3}=t,$$

which are the Cartesian equations of the straight line through the points A and B.]

The three points A, B, P are collinear; and if the linear equation (3) connecting their position vectors is written

$$(1-t)\mathbf{a}+t\mathbf{b}-\mathbf{r}=0,$$

with all the terms on one side, the algebraic sum of the coefficients of the vectors is zero. This is the necessary and sufficient condition that three points should be collinear. That it is a necessary condition has just been proved: for any point P collinear with A and B has a position vector given by (3) for some value of t. It is also sufficient; for, assuming the condition satisfied in a linear relation between $\mathbf{r}$, $\mathbf{a}$ and $\mathbf{b}$, we may make the coefficient of $\mathbf{r}$ unity and write the relation

$$\mathbf{r}=s\mathbf{b}+(1-s)\mathbf{a},$$

showing that P is a point on the straight line AB.

17. Bisector of the angle between two straight lines. To find the equation of the bisector of the angle between the straight lines OA and OB, parallel to the unit vectors $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ respectively, take the point O as origin, and let P be any point on the bisector.

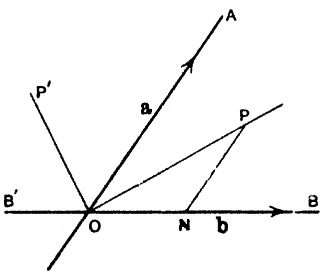

FIG. 17.

Then, if PN is drawn parallel to AO cutting OB in N, the angles OPN and NOP are equal, and $ON=NP$. But these are parallel to $\hat{\mathbf{b}}$ and $\hat{\mathbf{a}}$ respectively: so that $\overrightarrow{ON}=t\hat{\mathbf{b}}$ and $\overrightarrow{NP}=t\hat{\mathbf{a}}$, where t is some real number. The position vector of P is therefore

$$\mathbf{r}=t(\hat{\mathbf{a}}+\hat{\mathbf{b}}).$$

This is the required equation of the bisector, the value of t varying as P moves along the line.

The bisector $\overrightarrow{OP'}$ of the supplementary angle $B'OA$ is the bisector of the angle between straight lines whose directions are those of $\hat{\mathbf{a}}$ and $-\hat{\mathbf{b}}$; and its equation is therefore

$$\mathbf{r} = t(\hat{\mathbf{a}} - \hat{\mathbf{b}}).$$

18. The Triangle.

(i) *The internal bisector of the angle A of a triangle ABC divides the side BC in the ratio $AB : AC$.*

Let $\overrightarrow{AB} = \mathbf{c}$ and $AC = \mathbf{b}$. Then, with A as origin, the internal bisector of the angle A is the line

$$\mathbf{r} = t(\hat{\mathbf{b}} + \hat{\mathbf{c}}) = t\left(\frac{\mathbf{b}}{b} + \frac{\mathbf{c}}{c}\right)$$

$$= t\left(\frac{c\mathbf{b} + b\mathbf{c}}{bc}\right),$$

where, according to our usual notation, b is the module of $\mathbf{b}$, and $\hat{\mathbf{b}}$ the corresponding unit vector. Giving t the value $bc/(b+c)$ we see that

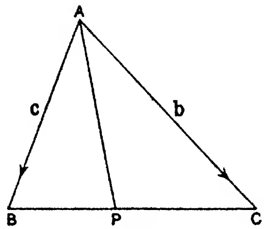

FIG. 18.

the bisector passes through the point $(c\mathbf{b} + b\mathbf{c})/(b+c)$, which is the centroid of the points B and C with associated numbers b and c respectively, and therefore lies in BC, dividing it in the ratio $c : b$ or $AB : AC$. Hence the theorem.

Similarly the external bisector of the angle A is the straight line

$$\mathbf{r} = t(\hat{\mathbf{c}} - \hat{\mathbf{b}}) = t\left(\frac{\mathbf{c}}{c} - \frac{\mathbf{b}}{b}\right)$$

$$= t\left(\frac{b\mathbf{c} - c\mathbf{b}}{bc}\right),$$

which passes through the point $(b\mathbf{c} - c\mathbf{b})/(b-c)$.

This point is the centroid of B and C with associated numbers* b and $-c$, and divides BC externally in the ratio $AB : AC$.

(ii) *The internal bisectors of the angles of a triangle are concurrent.*

The point P at which the internal bisector of the angle A cuts BC is the centroid of B and C with associated numbers b and c. The centroid of the three points A, B, C with associated numbers a, b, c therefore lies on AP and divides it in the ratio $(b+c) : a$. But by symmetry this point must also lie on the bisectors of the angles at B and C. Hence the bisectors are concurrent at this centroid.

Similarly it may be shown that the internal bisector of A and the external bisectors of B and C are concurrent at the centroid of the points A, B, C with associated numbers a, $-b$, $-c$ respectively.

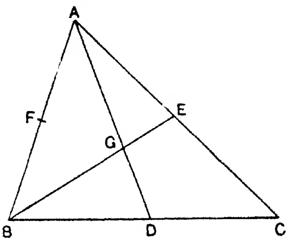

FIG 19

(iii) *The medians of a triangle are concurrent.*

Let D, E, F be the mid points of the sides of the triangle. Then D is the centroid of B and C; and the centroid of the three points A, B, C therefore lies on AD and divides it in the ratio $2 : 1$. And by symmetry this point also lies on the other medians, and is a common point of trisection.

It is well known that this point is also the centroid of area of the triangle ABC.

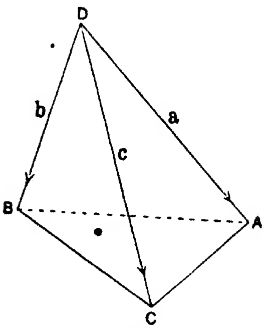

FIG. 20.

19. The Tetrahedron.

(i) *The joins of the mid points of opposite edges of a tetrahedron intersect and bisect each other.*

The mid point of DA is the centroid of D and A. The mid point of BC is the centroid of B and C. Hence the point of bisection of the line joining these mid points is the centroid of the four points A, B, C, D. And by symmetry this centroid is also the point of bisection of the lines joining the mid points of the other pairs of opposite sides. If

* The formula of Art. 9, defining the position of the centroid, is to apply whether the associated numbers are positive or negative.

a, **b**, **c** are the position vectors of A, B, C respectively relative to D, this common point of intersection is $\frac{1}{4}(\mathbf{a}+\mathbf{b}+\mathbf{c})$.

(ii) *The lines joining the vertices of a tetrahedron to the centroids of area of the opposite faces are concurrent.*

The centroid of area N of the face ABC is the centroid of the three points A, B, C. Hence the centroid of the four points A, B, C, D lies on AN and divides it in the ratio 3 : 1. And by symmetry this is also a point of quadrisection of the lines joining the other vertices to the centroids of area of the opposite faces.

It is well known that this point of intersection is also the centroid of volume of the tetrahedron.

Otherwise. The last two theorems may be proved by means of the vector equations of the straight lines, thus: Relative to D as origin the mid points of DA and BC are $\frac{1}{2}\mathbf{a}$ and $\frac{1}{2}(\mathbf{b}+\mathbf{c})$; and the equation of the straight line passing through these points is

$$\mathbf{r}=\frac{s\mathbf{a}}{2}+(1-s)\frac{(\mathbf{b}+\mathbf{c})}{2}.$$

Similarly the straight line through the mid points of DB and CA is

$$\mathbf{r}=\frac{t\mathbf{b}}{2}+(1-t)\frac{(\mathbf{c}+\mathbf{a})}{2}.$$

These straight lines will intersect if real values of s, t can be found which give identical values for $\mathbf{r}$. This requires the coefficients of like vectors to be equal in the two expressions; *i.e.*

$$s=1-t;\quad 1-s=t;\quad 1-s=1-t,$$

which are satisfied by $s=t=\frac{1}{2}$. Hence the lines intersect at the point $\frac{1}{4}(\mathbf{a}+\mathbf{b}+\mathbf{c})$.

Again the centroid of area of ABC is the point $\frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c})$; and the line joining D to this point has the equation

$$\mathbf{r}=s(\mathbf{a}+\mathbf{b}+\mathbf{c}).$$

Also the centroid of area of DAC is the point $\frac{1}{3}(\mathbf{a}+\mathbf{c})$; and the line joining this to B is

$$\mathbf{r}=t\mathbf{b}+(1-t)\frac{\mathbf{a}+\mathbf{c}}{3}.$$

These two lines intersect at the point for which $s=t=\frac{1}{4}$; that is the point $\frac{1}{4}(\mathbf{a}+\mathbf{b}+\mathbf{c})$. From the symmetry of this result the second theorem follows.

20. Vector equation of a plane.

(1) *To find the vector equation of the plane through the origin parallel to* **a** *and* **b**.

If P is any point on the plane its position vector $\overrightarrow{OP}$ is coplanar

with **a** and **b**, and may therefore be resolved into components parallel to these and expressed in the form

$$\mathbf{r} = s\mathbf{a} + t\mathbf{b}, \quad \text{.............................(1)}$$

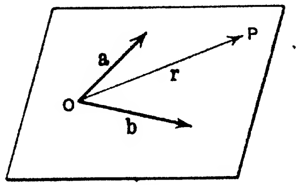

FIG. 21.

where s, t are numbers which vary as the point P moves over the plane. Any point on the plane is given by (1) for some values of s and t; and for all values of these variables the point $s\mathbf{a} + t\mathbf{b}$ lies on the given plane. We may therefore speak of (1) as the vector equation of the given plane.

(ii) *To find the vector equation of the plane through the point C parallel to* **a** *and* **b**.

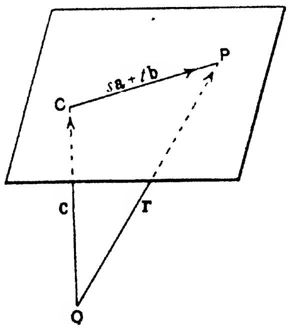

FIG. 22.

Let **c** be the position vector of C, and **r** that of any point P on the given plane. The vector $\overrightarrow{CP}$ is coplanar with **a** and **b**, and may therefore be written

$$\overrightarrow{CP} = s\mathbf{a} + t\mathbf{b},$$

as in the previous case.

Then
$$\mathbf{r} = \overrightarrow{OP} = \overrightarrow{OC} + \overrightarrow{CP}$$
$$= \mathbf{c} + s\mathbf{a} + t\mathbf{b}. \quad \text{.............................(2)}$$

This is the required equation to the plane, the numbers s, t varying as P moves over the plane. And no point off the plane can be represented by (2).

To find the equation of the plane through the three points A, B, C whose position vectors are $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, we observe that

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a} \quad \text{and} \quad \overrightarrow{AC} = \mathbf{c} - \mathbf{a};$$

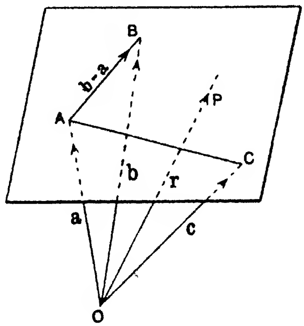

FIG. 23

so that the plane is one through A parallel to $\mathbf{b} - \mathbf{a}$ and $\mathbf{c} - \mathbf{a}$. Its equation is therefore

$$\mathbf{r} = \mathbf{a} + s(\mathbf{b} - \mathbf{a}) + t(\mathbf{c} - \mathbf{a})$$

or
$$\mathbf{r} = (1 - s - t)\mathbf{a} + s\mathbf{b} + t\mathbf{c}. \quad \text{.......................(3)}$$

The equations (1), (2) and (3) involve each two variable numbers s, t, and are analogous to the equations of the lines considered in Art. 16. *This form of the equation for a plane is not, however, the only form: nor is it the most convenient.* Another will be found in Art. 29 after the scalar product of two vectors has been defined.

The four points A, B, C, P in Fig. 23 are coplanar; and if the linear relation (3) connecting their position vectors is written

$$(1 - s - t)\mathbf{a} + s\mathbf{b} + t\mathbf{c} - \mathbf{r} = 0,$$

with all the terms on one side, the algebraic sum of the coefficients

of the vectors is zero. This is *the necessary and sufficient condition that four points should be coplanar.** That it is necessary has just been proved. It is also sufficient; for, assuming the condition satisfied in a linear relation between **r**, **a**, **b** and **c**, by making the coefficient of **r** unity, and denoting those of **b** and **c** by s and t, we obtain the relation in the form

$$\mathbf{r} = s\mathbf{b} + t\mathbf{c} + (1 - s - t)\mathbf{a},$$

showing that P is a point in the plane ABC.

21. As an example of the use of this form of the vector equation of a plane consider the following:

If any point O within a tetrahedron $ABCD$ is joined to the vertices, and AO, BO, CO, DO are produced to cut the opposite faces in P, Q, R, S respectively, then $\Sigma \dfrac{OP}{AP} = 1$.

With O as origin let the position vectors of A, B, C, D be **a**, **b**, **c**, **d** respectively. Any one of these vectors may be expressed in terms

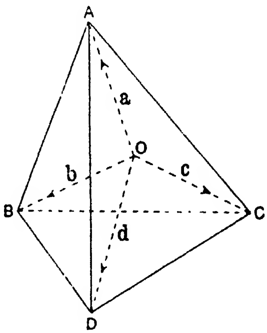

FIG. 24.

of the other three, so that there is a linear relation connecting them which may be written

$$l\mathbf{a} + m\mathbf{b} + n\mathbf{c} + p\mathbf{d} = 0. \quad \text{........................(1)}$$

The vector equation of the plane through B, C, D is

$$\mathbf{r} = s\mathbf{b} + t\mathbf{c} + (1 - s - t)\mathbf{d}, \quad \text{........................(2)}$$

while the line OP is $\mathbf{r} = -u\mathbf{a}$, where u is a variable number, positive for points of the line which lie on the opposite side of the origin

* An equivalent condition will be found in Art. 43.

to A. In virtue of the relation (1) we may write the equation of OP as

$$\mathbf{r} = \frac{u}{l}(m\mathbf{b} + n\mathbf{c} + p\mathbf{d}). \qquad \text{.........................(3)}$$

This line intersects the plane BCD at a point for which (2) and (3) give identical values for $\mathbf{r}$; and since $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ are non-coplanar vectors, this requires

$$s = \frac{mu}{l}; \quad t = \frac{nu}{l}; \quad 1 - s - t = \frac{pu}{l}.$$

From these, by addition, we have $u = l/(m+n+p)$, showing that the ratio

$$\frac{OP}{AP} = \frac{u}{1+u} = \frac{l}{l+m+n+p}.$$

The other ratios may be written down by cyclic permutation of the symbols, and their sum is obviously equal to unity.

22. Linear relation independent of the origin.

The necessary and sufficient condition that a linear relation, connecting the position vectors of any number of fixed points, should be independent of the origin, is that the algebraic sum of the coefficients is zero.

Let $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ be the position vectors of the fixed points $A_1, A_2, \ldots, A_n$ relative to an origin O, and let the linear relation be written

$$k_1\mathbf{a}_1 + k_2\mathbf{a}_2 + \ldots + k_n\mathbf{a}_n = 0. \qquad \text{.............................(1)}$$

Let O' be another point whose position vector is $\mathbf{l}$ (Fig. 10). Then if O' is taken as origin of vectors, the point A_m is $\mathbf{a}_m - \mathbf{l}$; and in order that the relation (1) should hold for the new origin also, we must have

$$k_1(\mathbf{a}_1 - \mathbf{l}) + k_2(\mathbf{a}_2 - \mathbf{l}) + \ldots + k_n(\mathbf{a}_n - \mathbf{l}) = 0,$$

which, in virtue of (1), reduces to

$$k_1 + k_2 + \ldots + k_n = 0. \qquad \text{.......................(2)}$$

The condition is therefore *necessary*. It is also *sufficient*; for if (2) holds, each vector in (1) may be diminished by the same constant vector $\mathbf{l}$, and the equation will still be true. Hence the relation will hold with O' as origin.

23. Vector areas. Consider the type of vector quantity whose magnitude is an area. Such a quantity is associated with each plane figure, the magnitude being the area of the figure, and the direction that of the normal to the plane of the figure. This

vector area therefore specifies both the area and the orientation of the plane figure. But as the direction might be either of two opposite directions along the normal, some convention is necessary. The area clearly has no sign in itself, and can be regarded as positive or negative only with reference to the direction in which the boundary of the figure is described, or the side of the plane from which it is viewed.

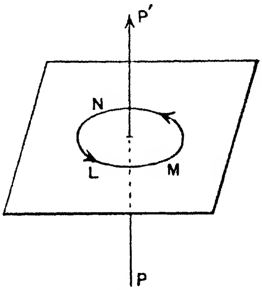

FIG. 25.

Consider the area of the figure bounded by the closed curve LMN, which is regarded as being traced out in the direction of the arrows. The normal vector $\overrightarrow{PP'}$ bears to this direction of rotation the same relation as the translation to the direction of rotation of a right-handed screw. The area LMN is regarded as positive relative to the direction of $\overrightarrow{PP'}$.

With this convention a vector area may be represented by a vector normal to the plane of the figure, in the direction relative to which it is positive, and with module equal to the measure of the area. *The sum of two vector areas represented by* $\mathbf{a}$ *and* $\mathbf{b}$ *is defined to be the vector area represented by* $\mathbf{a}+\mathbf{b}$.

It is a well-known geometrical result that the area A' of the orthogonal projection of a plane figure of area A, upon another plane inclined at an angle θ to it, is $A \cos \theta$. If then $\mathbf{a}$, $\mathbf{a}'$ are the vectors representing the vector areas A, A' respectively, $\mathbf{a}'$ is the resolute of $\mathbf{a}$ in the direction normal to the plane of projection: for $a' = a \cos \theta$. If there are several figures of areas $A, B, \ldots$ in planes inclined at angles $\theta, \phi, \ldots$ respectively to a given plane, the sum of the areas of their projections on this plane is

$$A \cos \theta + B \cos \phi + \ldots,$$

and, as a vector area, is represented by the sum of the resolutes of $\mathbf{a}, \mathbf{b}, \ldots$ perpendicular to the plane of projection. Thus the sum of the areas of the projections is equal to the projection of the sum of the vector areas.

When the areas are those of the faces of a solid figure they are regarded as positive relative to the outward drawn normals. Hence the vector areas of the faces will be represented by vectors in the directions of the normals drawn outward.

Theorem. *The sum of the vector areas of the faces of a closed polyhedron is zero.*

Consider the areas of the projections of the faces on any fixed plane. Some of these are positive and some negative: for some of the faces have their outward normals directed from the plane, and others toward the plane. But the sum of the areas of the projections of the former is equal in magnitude to that of the latter; for each sum is equal to the area enclosed by the projection of the (skew) polygon composed of those edges of the polyhedron which separate the former faces from the latter. Hence the sum of the areas of the projections of all the faces is zero; and therefore the projection of the sum of the vector areas is zero. And since this is true for any plane of projection, the sum of the vector areas is identically zero.

EXERCISES ON CHAPTER II.

Prove the following by *vector* methods:

√ 1. The diagonals of a parallelogram bisect each other. Conversely, if the diagonals of a quadrilateral bisect each other it is a parallelogram.

√ 2. The join of the mid points of two sides of a triangle is parallel to the third side, and of half its length.

3. The four diagonals of a parallelepiped, and the joins of the mid points of opposite edges, are concurrent at a common point of bisection.

4. In a skew quadrilateral the joins of the mid points of opposite edges bisect each other. Also, if the mid points of the sides be joined in order, the figure so formed is a parallelogram.

5. Show that the sum of the three vectors determined by the medians of a triangle directed from the vertices is zero.

√ 6. The three points whose position vectors are $\mathbf{a}$, $\mathbf{b}$ and $3\mathbf{a}-2\mathbf{b}$ are collinear.

7. *ABCD* is a parallelogram and *O* the point of intersection of its diagonals. Show that for any origin (not necessarily in the

plane of the figure) the sum of the position vectors of the vertices is equal to four times that of O.

8. What is the vector equation of the straight line through the points $\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $3\mathbf{k} - 2\mathbf{j}$?

Find where this line cuts the plane through the origin and the points $4\mathbf{j}$ and $2\mathbf{i} + \mathbf{k}$.

9. No two lines, joining points in two non-coplanar straight lines, can be parallel.

10. If M, N are the mid points of the sides AB, CD of a parallelogram $ABCD$, DM and BN cut the diagonal AC at its points of trisection, which are also points of trisection of DM and BN respectively.

11. Three concurrent straight lines OA, OB, OC are produced to D, E, F respectively. Prove that the points of intersection of AB and DE, BC and EF, CA and FD are collinear.

12. Prove the converse of the last exercise—that if the points of intersection are real and collinear, then DA, EB, FC are concurrent.

13. The internal bisector of one angle of a triangle, and the external bisectors of the other two, are concurrent.

14. The mid points of the six edges of a cube which do not meet a particular diagonal are coplanar.

15. The six planes which contain one edge and bisect the opposite edge of a tetrahedron meet in a point.

16. If two parallel planes are cut by a third plane the lines of intersection are parallel.

17. If a tetrahedron is cut by a plane parallel to two opposite edges the section is a parallelogram.

18. If a straight line is drawn parallel to the base of a triangle, the line joining the vertex to the intersection of the diagonals of the trapezium so formed bisects the base of the triangle.

19. The straight lines through the mid points of three coplanar edges of a tetrahedron, each parallel to the line joining a fixed point O to the mid point of the opposite edge, are concurrent at a point P, such that OP is bisected by the centroid (of volume) of the tetrahedron.

20. Using the vector equation of a straight line, show that the mid points of the diagonals of a complete quadrilateral are collinear; and also establish the harmonic property of the figure.

CHAPTER III.

PRODUCTS OF TWO VECTORS. THE PLANE AND THE SPHERE.

24. From the nature of a vector, as a quantity involving direction as well as magnitude, it is impossible to say *à priori* what the product of two vectors ought to be. But by examining the ways in which two vector quantities enter into combination in Physics and Mechanics we are led to define two distinct kinds of products, the one a number and the other a vector. Each of these products is jointly proportional to the modules of the two vectors; and each follows the distributive law, that the product of $\mathbf{a}$ with $\mathbf{b}+\mathbf{c}$ is equal to the sum of the products of $\mathbf{a}$ with $\mathbf{b}$ and of $\mathbf{a}$ with $\mathbf{c}$. We borrow from algebra the term *multiplication* for the process by which a product is formed from two vectors; and these are called the *factors* of the product.

The Scalar or Dot Product.

25. Scalar quantities are of frequent occurrence, which depend each upon two vector quantities in such a way as to be jointly proportional to their magnitudes and to the cosine of their mutual inclination. An example of such is the work done by a force during a displacement of the body acted upon. We therefore find it convenient to adopt the following

Definition. *The scalar product of two vectors* $\mathbf{a}$ *and* $\mathbf{b}$, *whose directions are inclined at an angle* θ, *is the real number* $ab\cos\theta$, *and is written* *

$$\mathbf{a}\cdot\mathbf{b} = ab\cos\theta = \mathbf{b}\cdot\mathbf{a}.$$

* An alternative notation for the scalar product is $(\mathbf{ab})$.

The order of the factors may be reversed without altering the value of the product. Further, $b \cos \theta$ is the measure of the length of the resolute of **b** in the direction of **a**, positive or negative according as θ is acute or obtuse. Hence $\mathbf{a} \cdot \mathbf{b}$ is the product of two numbers, which measure the length of one of the vectors and the length of the resolute of the other in its direction.

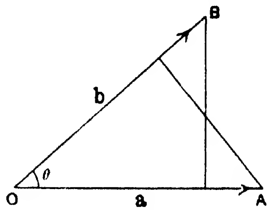

FIG. 26

If two vectors **a**, **b** are perpendicular, $\cos \theta = 0$, and their scalar product is zero. Hence the condition of perpendicularity of two finite vectors is expressed by

$$\mathbf{a} \cdot \mathbf{b} = 0.$$

If the vectors have the same direction, $\cos \theta = 1$, and $\mathbf{a} \cdot \mathbf{b} = ab$. If their directions are opposite, $\cos \theta = -1$, and $\mathbf{a} \cdot \mathbf{b} = -ab$. The scalar product of any two *unit* vectors is equal to the cosine of the angle between their directions.

When the factors are equal vectors their scalar product $\mathbf{a} \cdot \mathbf{a}$ is called the *square* of **a**, and is written $\mathbf{a}^2$. Thus

$$\mathbf{a}^2 = \mathbf{a} \cdot \mathbf{a} = a^2,$$

the square of a vector being thus equal to the square of its module. The square of any unit vector is unity. In particular,

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = 1,$$

but since these vectors are mutually perpendicular

$$\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0. \checkmark$$

These relations will be constantly employed.

If either factor is multiplied by a number, the scalar product is multiplied by that number. For

$$(n\mathbf{a}) \cdot \mathbf{b} = nab \cos \theta = \mathbf{a} \cdot (n\mathbf{b}).$$

Since the scalar product is a number, it may occur as the numerical coefficient of a vector. Thus $\mathbf{a} \cdot \mathbf{b}\mathbf{c} \equiv (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$ is a vector in the direction of **c** with module $\mathbf{a} \cdot \mathbf{b}c$. The combination $\mathbf{a} \cdot \mathbf{b}\,\mathbf{c} \cdot \mathbf{d}$ of four vectors is simply the product of the two numbers $\mathbf{a} \cdot \mathbf{b}$ and $\mathbf{c} \cdot \mathbf{d}$.

If $\mathbf{r}$ is inclined at an angle ϕ to $\mathbf{a}$, the resolute or resolved part of $\mathbf{r}$ in the direction of $\mathbf{a}$ is

$$r\cos\phi\,\hat{\mathbf{a}} = \frac{ar\cos\phi}{a^2}\,\mathbf{a} = \frac{\mathbf{a}\cdot\mathbf{r}}{a^2}\,\mathbf{a}.$$

Hence if $\mathbf{r}$ is resolved into two components in the plane of $\mathbf{a}$ and $\mathbf{r}$, one parallel and the other perpendicular to $\mathbf{a}$, these components are

$$\frac{\mathbf{a}\cdot\mathbf{r}}{a^2}\,\mathbf{a} \quad \text{and} \quad \mathbf{r} - \frac{\mathbf{a}\cdot\mathbf{r}}{a^2}\,\mathbf{a}$$

respectively.

Similarly, if any vector $\mathbf{r}$ is resolved into components parallel to the unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, these components are $\mathbf{i}\cdot\mathbf{r}\mathbf{i}$, $\mathbf{j}\cdot\mathbf{r}\mathbf{j}$, $\mathbf{k}\cdot\mathbf{r}\mathbf{k}$ respectively, and $\mathbf{r}$ is their sum. Thus

$$\mathbf{r} = \mathbf{i}\cdot\mathbf{r}\mathbf{i} + \mathbf{j}\cdot\mathbf{r}\mathbf{j} + \mathbf{k}\cdot\mathbf{r}\mathbf{k}.$$

26. Distributive Law. It is easy to show that the distributive law of multiplication holds for scalar products; that is,

$$\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c}.$$

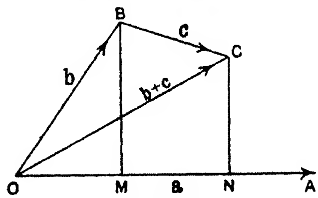

FIG. 27.

Let $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{BC}$ be equal to $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ respectively. Project OB, BC on OA. Then the length of the projection of OC is the algebraic sum of the lengths of the projections of OB and BC. Therefore

$$\begin{aligned}\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) &= \mathbf{a}\cdot(\overrightarrow{OC}) = a\,.\,ON\\ &= a(OM+MN) = a\,.\,OM + a\,.\,MN\\ &= \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c}.\end{aligned}$$

Repeated application of this result shows that the scalar product of two sums of vectors may be expanded as in ordinary algebra. Thus

$$\begin{aligned}(\mathbf{a}+\mathbf{b}+\ldots)\cdot(\mathbf{l}+\mathbf{m}+\ldots) = {}&\mathbf{a}\cdot\mathbf{l} + \mathbf{a}\cdot\mathbf{m} + \ldots\\ &+\mathbf{b}\cdot\mathbf{l} + \mathbf{b}\cdot\mathbf{m} + \ldots\\ &+\ldots\end{aligned}$$

In particular, $(\mathbf{a}+\mathbf{b})^2=\mathbf{a}^2+2\mathbf{a}\cdot\mathbf{b}+\mathbf{b}^2,$

while $(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})=\mathbf{a}^2-\mathbf{b}^2.$

As another useful particular case, suppose that two vectors **a**, **b** are expressed in terms of the unit vectors **i**, **j**, **k**. Then, with the usual notation,

$$\begin{aligned}\mathbf{a}\cdot\mathbf{b}&=(a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})\cdot(b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k})\\&=a_1b_1+a_2b_2+a_3b_3,\end{aligned}$$

since the scalar product of two perpendicular vectors is zero. And because the direction cosines l, m, n of **a** are a_1/a, a_2/a, a_3/a, and those l', m', n' of **b** are b_1/b, b_2/b, b_3/b, it follows that

$$\cos\theta=ll'+mm'+nn',$$

where θ is the angle between the directions of **a** and **b**.

Examples.

(1) *Particles of masses m_1, m_2, m_3, ... are placed at the points A, B, C, ... respectively, and G is their c.m. Prove that for any point P,*

$$m_1AP^2+m_2BP^2+\ldots=m_1AG^2+m_2BG^2+\ldots+(\Sigma m)PG^2.$$

Take G as origin, and let **s** be the position vector of P, and $\mathbf{r}_1$, $\mathbf{r}_2$, ... those of A, B, ... respectively. Then

$$\Sigma m_1\mathbf{r}_1=0.$$

Also $\overrightarrow{AP}=\mathbf{s}-\mathbf{r}_1$, so that

$$\begin{aligned}\Sigma m_1AP^2&=\Sigma m_1(\mathbf{s}-\mathbf{r}_1)^2\\&=\Sigma m_1\mathbf{s}^2-2\mathbf{s}\cdot(\Sigma m_1\mathbf{r}_1)+\Sigma m_1\mathbf{r}_1^2\\&=(\Sigma m_1)PG^2+\Sigma m_1AG^2.\end{aligned}$$

(2) *In a tetrahedron, if two pairs of opposite edges are perpendicular, the third pair are also perpendicular to each other; and the sum of the squares on two opposite edges is the same for each pair.*

Using the notation of Fig. 20 we have $\overrightarrow{AB}=\mathbf{b}-\mathbf{a}$, $\overrightarrow{AC}=\mathbf{c}-\mathbf{a}$ and $\overrightarrow{CB}=\mathbf{b}-\mathbf{c}$. Hence if BD is perpendicular to CA,

$$\mathbf{b}\cdot(\mathbf{c}-\mathbf{a})=0,$$

that is $\mathbf{b}\cdot\mathbf{c}=\mathbf{a}\cdot\mathbf{b}.$

And similarly, if DA is perpendicular to BC,

$$\mathbf{a}\cdot(\mathbf{b}-\mathbf{c})=0,$$

that is $\mathbf{a}\cdot\mathbf{b}=\mathbf{c}\cdot\mathbf{a}.$

Thus $$\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{c}=\mathbf{c}\cdot\mathbf{a}, \quad \text{.........................(i)}$$
whence $$\mathbf{c}\cdot(\mathbf{b}-\mathbf{a})=0,$$
showing that DC is perpendicular to BA.

Further, the sum of the squares on BD and CA is

$$\mathbf{b}^2+(\mathbf{c}-\mathbf{a})^2=\mathbf{b}^2+\mathbf{c}^2+\mathbf{a}^2-2\mathbf{a}\cdot\mathbf{c},$$

and, in virtue of (i), this is the same as in the other two cases.

(3) *Prove that, if θ_{mn} is the mutual inclination of the m^{th} and n^{th} of any four non-coplanar straight lines,*

$$\begin{vmatrix} 1 & \cos\theta_{12} & \cos\theta_{13} & \cos\theta_{14} \\ \cos\theta_{21} & 1 & \cos\theta_{23} & \cos\theta_{24} \\ \cos\theta_{31} & \cos\theta_{32} & 1 & \cos\theta_{34} \\ \cos\theta_{41} & \cos\theta_{42} & \cos\theta_{43} & 1 \end{vmatrix}=0.$$

Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ be four unit vectors parallel to the given lines. Any one of these may be expressed linearly in terms of the others, so that there is a linear relation between them which may be written

$$l\mathbf{a}+m\mathbf{b}+n\mathbf{c}+p\mathbf{d}=0. \quad \text{...............................(i)}$$

Form the scalar product of each side with $\mathbf{a}$. Then, since $\mathbf{a}\cdot\mathbf{b}=\cos\theta_{12}$, and so on, we have

$$l\mathbf{a}^2+m\mathbf{a}\cdot\mathbf{b}+n\mathbf{a}\cdot\mathbf{c}+p\mathbf{a}\cdot\mathbf{d}=0$$

or $$l+m\cos\theta_{12}+n\cos\theta_{13}+p\cos\theta_{14}=0.$$

Three similar equations may be found on forming the scalar product of (i) with $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ successively. Then, eliminating l, m, n, p from these four equations, we have the required result.

(4) *Oblique coordinate axes.* Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be unit vectors in the positive directions of the axes, and λ, μ, ν their mutual inclinations, so that $$\mathbf{b}\cdot\mathbf{c}=\cos\lambda, \quad \mathbf{c}\cdot\mathbf{a}=\cos\mu, \quad \mathbf{a}\cdot\mathbf{b}=\cos\nu.$$

If x, y, z are the oblique coordinates of a point P its position vector is

$$\mathbf{r}=x\mathbf{a}+y\mathbf{b}+z\mathbf{c}.$$

The square of its distance from the origin is

$$\begin{aligned}\mathbf{r}^2&=(x\mathbf{a}+y\mathbf{b}+z\mathbf{c})^2\\&=x^2+y^2+z^2+2yz\cos\lambda+2zx\cos\mu+2xy\cos\nu.\end{aligned}$$

The square of the distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is got by putting x_1-x_2 for x, etc., in the above expression.

The angle α which $\overrightarrow{OP}$ makes with the x-axis is given by

$$\cos\alpha=\frac{\mathbf{a}\cdot\mathbf{r}}{|\mathbf{r}|}=\frac{x+y\cos\nu+z\cos\mu}{\sqrt{x^2+y^2+z^2+2yz\cos\lambda+\dots}},$$

and if P' is the point x', y', z' the angle ϕ between $\overrightarrow{OP}$ and $\overrightarrow{OP'}$ is given by

$$\cos\phi = \frac{\mathbf{r}\cdot\mathbf{r}'}{rr'},$$

whose value is easily written down.

The Vector or Cross Product.

27. Vector quantities are of frequent occurrence, which depend each upon two other vector quantities in such a way as to be jointly proportional to their magnitudes and to the sine of their mutual inclination, and to have a direction perpendicular to each of them. We are therefore led to adopt the following

Definition. *The vector product of two vectors* **a** *and* **b**, *whose directions are inclined at an angle* θ, *is the vector whose module is* $ab\sin\theta$, *and whose direction is perpendicular to both* **a** *and* **b**, *being positive relative to a rotation from* **a** *to* **b**. (Arts. 7, 23.)

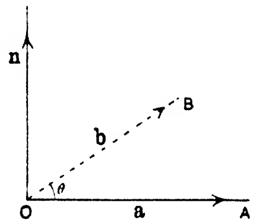

FIG. 28

We write * it $\mathbf{a}\times\mathbf{b}$, so that

$$\mathbf{a}\times\mathbf{b} = ab\sin\theta\,\mathbf{n},$$

where $\mathbf{n}$ is a unit vector perpendicular to the plane of **a**, **b**, having the same direction as the translation of a right-handed screw due to a rotation from **a** to **b**. From this it follows that $\mathbf{b}\times\mathbf{a}$ has the opposite direction to $\mathbf{a}\times\mathbf{b}$, but the same length, so that

$$\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}.$$

The order of the factors in a vector product is not commutative; for a reversal of the order alters the sign of the product.

Consider the parallelogram $OAPB$ whose sides OA, OB have the lengths and directions of **a** and **b** respectively. The area of the figure is $ab\sin\theta$, and the vector area $OAPB$, whose boundary is described in this sense, is represented by $ab\sin\theta\,\mathbf{n} = \mathbf{a}\times\mathbf{b}$. This simple geometrical relation will be found useful. The vector area $OBPA$ is of course represented by $\mathbf{b}\times\mathbf{a}$.

For two parallel vectors $\sin\theta$ is zero, and their vector product vanishes. The relation $\mathbf{a}\times\mathbf{b} = 0$ is thus the condition of parallelism

* An alternative notation for the vector product is $[\mathbf{ab}]$.

of two finite vectors. In particular, $\mathbf{r} \times \mathbf{r} = 0$ is true for all vectors.

If, however, $\mathbf{a}$ and $\mathbf{b}$ are perpendicular, $\mathbf{a} \times \mathbf{b}$ is a vector whose module is ab, and whose direction is such that $\mathbf{a}$, $\mathbf{b}$, $\mathbf{a} \times \mathbf{b}$ form a right-handed system of mutually perpendicular vectors.

If $\mathbf{a}$, $\mathbf{b}$ are both *unit* vectors the module of $\mathbf{a} \times \mathbf{b}$ is the sine of their angle of inclination. For the particular unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ we have

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0,$$

while

$$\mathbf{i} \times \mathbf{j} = \mathbf{k} = -\mathbf{j} \times \mathbf{i},$$
$$\mathbf{j} \times \mathbf{k} = \mathbf{i} = -\mathbf{k} \times \mathbf{j},$$
$$\mathbf{k} \times \mathbf{i} = \mathbf{j} = -\mathbf{i} \times \mathbf{k}.$$

These relations will be constantly employed.

If either factor is multiplied by a number, their product is multiplied by that number. For

$$(m\mathbf{a}) \times \mathbf{b} = mab \sin\theta \, \mathbf{n} = \mathbf{a} \times (m\mathbf{b}).$$

28. The Distributive Law. We shall now show that the distributive law holds for vector products also; that is

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}.$$

Suppose first that the vectors are not coplanar, and consider the triangular prism whose three parallel edges have the length and direction of $\mathbf{a}$, and whose parallel ends PQR, $P'Q'R'$ are triangular with $\overrightarrow{PQ} = \mathbf{b}$, $\overrightarrow{QR} = \mathbf{c}$, and therefore $\overrightarrow{PR} = \mathbf{b} + \mathbf{c}$. The sum of the vector areas of the faces of this closed polyhedron is zero (Art. 23). But these are represented by the outward drawn normal vectors $\frac{1}{2}\mathbf{c} \times \mathbf{b}$ and $\frac{1}{2}\mathbf{b} \times \mathbf{c}$ for the triangular ends, and $\mathbf{b} \times \mathbf{a}$, $\mathbf{c} \times \mathbf{a}$, $\mathbf{a} \times (\mathbf{b} + \mathbf{c})$ for the other faces. Of these five vectors the first two are equal in length but opposite in direction. Hence the sum of the other three must vanish identically; that is

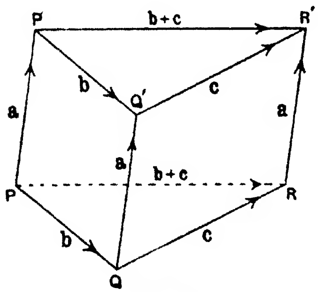

FIG. 28.

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) + \mathbf{b} \times \mathbf{a} + \mathbf{c} \times \mathbf{a} = 0,$$

which is equivalent to

$$\mathbf{a}\times(\mathbf{b}+\mathbf{c})=\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}.$$

On changing the sign of each term we may also write the relation

$$(\mathbf{b}+\mathbf{c})\times\mathbf{a}=\mathbf{b}\times\mathbf{a}+\mathbf{c}\times\mathbf{a}.$$

This proves the distributive law for non-coplanar vectors. It is needless to remark that the order of the factors must be maintained in each term; for a change of order is equivalent to a change of sign.

If the vectors are coplanar Fig. 29 may be regarded as a plane figure. The triangles PQR, $P'Q'R'$ are congruent, and therefore the sum of the areas of the parallelograms $PQQ'P'$, $Q'QRR'$ is equal to that of $PRR'P'$. Hence the relation

$$\mathbf{b}\times\mathbf{a}+\mathbf{c}\times\mathbf{a}=(\mathbf{b}+\mathbf{c})\times\mathbf{a}.$$

Repeated application of this result shows that the vector product of two sums of vectors may be expanded as in ordinary algebra, provided the order of the factors is maintained in each term. Thus

$$\begin{aligned}(\mathbf{a}+\mathbf{b}+\ldots)\times(\mathbf{l}+\mathbf{m}+\ldots)=&\,\mathbf{a}\times\mathbf{l}+\mathbf{a}\times\mathbf{m}+\ldots\\ &+\mathbf{b}\times\mathbf{l}+\mathbf{b}\times\mathbf{m}+\ldots\\ &+\ldots.\end{aligned}$$

As a useful particular case consider two vectors $\mathbf{a}$, $\mathbf{b}$ expressed in terms of their rectangular components. Then, with the usual notation,

$$\begin{aligned}\mathbf{a}\times\mathbf{b}&=(a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})\times(b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k})\\ &=(a_2b_3-a_3b_2)\mathbf{i}+(a_3b_1-a_1b_3)\mathbf{j}+(a_1b_2-a_2b_1)\mathbf{k},\end{aligned}$$

in virtue of the relations proved in the preceding Art. This result is very important. It may be written in the determinantal form

$$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ \mathbf{i} & \mathbf{j} & \mathbf{k}\end{vmatrix}.$$

This vector has a module $ab\sin\theta$. Hence, on squaring both members of the above equation and dividing by a^2b^2, we find for the *sine of the angle between two vectors* $\mathbf{a}$ *and* $\mathbf{b}$,

$$\sin^2\theta=\frac{(a_2b_3-a_3b_2)^2+(a_3b_1-a_1b_3)^2+(a_1b_2-a_2b_1)^2}{(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)}.$$

If l, m, n and l', m', n' are the direction cosines of **a** and **b** respectively,* this is equivalent to

$$\sin^2\theta = (mn' - m'n)^2 + (nl' - n'l)^2 + (lm' - l'm)^2.$$

It is worth noticing that if $\mathbf{b} = \mathbf{c} + n\mathbf{a}$, where n is any real number, then

$$\mathbf{a}\times\mathbf{b} = \mathbf{a}\times(\mathbf{c} + n\mathbf{a}) = \mathbf{a}\times\mathbf{c}.$$

Conversely, if $\mathbf{a}\times\mathbf{b} = \mathbf{a}\times\mathbf{c}$ it does not follow that $\mathbf{b} = \mathbf{c}$, but that **b** differs from **c** by some vector parallel to **a**, which may or may not be zero.

Geometry of the Plane.

29. Equation of a plane. Let p be the length of the perpendicular ON from the origin O to the given plane, and $\hat{\mathbf{n}}$

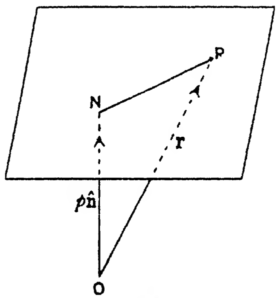

FIG. 30.

the unit vector normal to the plane, having the direction O to N. Then $\overrightarrow{ON} = p\hat{\mathbf{n}}$. If **r** is the position vector of any point P on the plane, $\mathbf{r}\cdot\hat{\mathbf{n}}$ is the projection of OP on ON, and is therefore equal to p. Thus the equation

$$\mathbf{r}\cdot\hat{\mathbf{n}} = p$$

is satisfied by the position vector of every point on the plane; and clearly no point off the plane satisfies this relation. We may therefore speak of it as the equation of the given plane. If **n** is any vector parallel to $\hat{\mathbf{n}}$, and therefore normal to the plane, the equation may be written

$$\mathbf{r}\cdot\mathbf{n} = np = q \text{ (say)}, \qquad (1)$$

which we shall take as the standard form for the equation of a

plane. If x, y, z are the coordinates of P referred to rectangular axes through O, and l, m, n are the direction cosines of $\mathbf{n}$, the equation (1) is equivalent to

$$lx + my + nz = p,$$

which is the form used in coordinate geometry.

Conversely, every equation of the form (1) represents a plane. For if $\mathbf{r}_1$, $\mathbf{r}_2$ are two position vectors satisfying the equation, and m, m' any real numbers, the vector $(m\mathbf{r}_1 + m'\mathbf{r}_2)/(m + m')$ also satisfies it. That is to say, if P_1, P_2 are two points on the surface represented by the equation, any point in the straight line P_1P_2 lies on the surface, which must therefore be a plane.

Consider the *plane through the point* $\mathbf{d}$ *and perpendicular to* $\mathbf{n}$. If $\mathbf{r}$ is any point on it, the vector $\mathbf{r} - \mathbf{d}$ is parallel to the plane, and therefore perpendicular to $\mathbf{n}$. Thus

$$(\mathbf{r} - \mathbf{d})\cdot\mathbf{n} = 0$$

or

$$\mathbf{r}\cdot\mathbf{n} = \mathbf{d}\cdot\mathbf{n}. \qquad \text{.........(2)}$$

This is of the form (1), and is the equation of the plane considered. The length of the perpendicular from the origin to the plane is $\mathbf{d}\cdot\mathbf{n}/n$. It is equal to the projection of OD on the normal to the plane, D being the point whose position vector is $\mathbf{d}$.

The *angle between two planes* whose equations are

$$\mathbf{r}\cdot\mathbf{n} = q,$$
$$\mathbf{r}\cdot\mathbf{n}' = q',$$

is equal to the angle θ between their normals. But

$$\mathbf{n}\cdot\mathbf{n}' = nn' \cos\theta,$$

so that the required angle is

$$\theta = \cos^{-1}\frac{\mathbf{n}\cdot\mathbf{n}'}{nn'}.$$

To find the *intercepts on the coordinate axes* made by the plane $\mathbf{r}\cdot\mathbf{n} = q$, we observe that if x_1 is the length of the intercept on the x-axis the point $x_1\mathbf{i}$ lies on the plane, and therefore $x_1\mathbf{i}\cdot\mathbf{n} = q$; showing that

$$x_1 = \frac{q}{\mathbf{i}\cdot\mathbf{n}}.$$

Similarly, the lengths of the intercepts on the other axes are

$$y_1 = \frac{q}{\mathbf{j}\cdot\mathbf{n}} \quad \text{and} \quad z_1 = \frac{q}{\mathbf{k}\cdot\mathbf{n}}.$$

30. Distance of a point from a plane. We have seen that the length p of the perpendicular ON from the origin to the plane $\mathbf{r}\cdot\mathbf{n}=q$ is q/n. Suppose we require the *perpendicular distance of any other point P' from the plane.* Consider the parallel plane through P'. The length p' of the perpendicular from O to this plane is $\mathbf{r}'\cdot\mathbf{n}/n$, where $\mathbf{r}'$ is the position vector of P'. Therefore the length of the perpendicular from P' to the given plane is

$$p-p'=\frac{q-\mathbf{r}'\cdot\mathbf{n}}{n}.$$

The vector $\overrightarrow{P'M}$ determined by this perpendicular is

$$\frac{(q-\mathbf{r}'\cdot\mathbf{n})}{n^2}\mathbf{n}.$$

In deducing the above formula we have considered the distance of P' from the plane as positive when measured in the direction of $\mathbf{n}$, that is from O to N. The expression found will therefore be positive for points on the same side of the plane as the origin : negative for points on the opposite side.

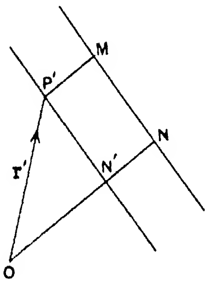

FIG. 31.

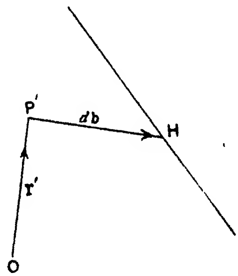

FIG. 32.

To find the *distance of P' from the plane measured in the direction of the unit vector* $\mathbf{b}$, let a parallel to $\mathbf{b}$ drawn through P' cut the plane in H. Then if d is the length of $P'H$, the position vector of H is $\mathbf{r}'+d\mathbf{b}$; and since this point lies on the given plane,

$$(\mathbf{r}'+d\mathbf{b})\cdot\mathbf{n}=q,$$

giving
$$d=\frac{q-\mathbf{r}'\cdot\mathbf{n}}{\mathbf{b}\cdot\mathbf{n}}.$$

This is the required distance.

The *equations of the planes bisecting the angles between two given planes*

$$\mathbf{r}\cdot\mathbf{n} = q,$$
$$\mathbf{r}\cdot\mathbf{n}' = q',$$

are obtained from the fact that a point on either bisector is equidistant from the two planes. Since we consider perpendicular distance as positive when measured in the direction from the origin to the plane, for points on the plane bisecting the angle in which the origin lies, the perpendicular distances will have the same sign; but for points on the other bisector, opposite signs. Hence if $\mathbf{n}$, $\mathbf{n}'$ are directed from the origin toward the plane in each case, for any point $\mathbf{r}'$ on the bisector of the angle in which O lies we must have

$$\frac{q - \mathbf{r}'\cdot\mathbf{n}}{n} = \frac{q' - \mathbf{r}'\cdot\mathbf{n}'}{n'}.$$

And since this relation is satisfied by every point on the bisector, the equation of this plane is

$$\mathbf{r}\cdot(\hat{\mathbf{n}} - \hat{\mathbf{n}}') = \frac{q}{n} - \frac{q'}{n'}.$$

Similarly the equation of the other bisector is

$$\mathbf{r}\cdot(\hat{\mathbf{n}} + \hat{\mathbf{n}}') = \frac{q}{n} + \frac{q'}{n'}.$$

√ **31. Planes through the intersection of two planes.** Consider two given planes

$$\left.\begin{aligned}\mathbf{r}\cdot\mathbf{n} &= q,\\ \mathbf{r}\cdot\mathbf{n}' &= q'.\end{aligned}\right\} \quad\text{.......................................}\quad (1)$$

Then the equation $\quad \mathbf{r}\cdot(\mathbf{n} - \lambda\mathbf{n}') = q - \lambda q'$ .... (2)

is that of a plane perpendicular to $\mathbf{n} - \lambda\mathbf{n}'$. And if any vector $\mathbf{r}$ satisfies each of the equations (1), it also satisfies this. Hence any point on the line of intersection of the given planes lies on the plane represented by (2), which is therefore the equation of a plane through this line. And by giving a suitable real value to λ we can make it represent any plane through the intersection; for the normal to the plane, viz. $\mathbf{n} - \lambda\mathbf{n}'$, can be given any direction in the plane of $\mathbf{n}$ and $\mathbf{n}'$. For instance the plane may be made to pass through any point $\mathbf{d}$ by choosing λ so that

$$\mathbf{d}\cdot(\mathbf{n} - \lambda\mathbf{n}') = q - \lambda q',$$

that is

$$\lambda = \frac{\mathbf{d}\cdot\mathbf{n} - q}{\mathbf{d}\cdot\mathbf{n}' - q'}.$$

The *equation of the line of intersection of the planes* (1) may be found as follows. This line is perpendicular to both $\mathbf{n}$ and $\mathbf{n}'$, and is therefore parallel to $\mathbf{n}\times\mathbf{n}'$. The perpendicular ON from the origin to the line is parallel to the plane of $\mathbf{n}$ and $\mathbf{n}'$, so that

$$\overrightarrow{ON} = l\mathbf{n} + l'\mathbf{n}',$$

where l, l' are real numbers. But since N lies on both planes, this vector must satisfy both the equations (1), *i.e.*

$$(l\mathbf{n} + l'\mathbf{n}')\cdot\mathbf{n} = q,$$
$$(l\mathbf{n} + l'\mathbf{n}')\cdot\mathbf{n}' = q'.$$

From these two equations the values of l and l' are determined as

$$l = \frac{qn'^2 - q'\mathbf{n}\cdot\mathbf{n}'}{n^2n'^2 - (\mathbf{n}\cdot\mathbf{n}')^2}; \quad l' = \frac{q'n^2 - q\mathbf{n}\cdot\mathbf{n}'}{n^2n'^2 - (\mathbf{n}\cdot\mathbf{n}')^2}.$$

The line of intersection of the planes is a line through N parallel to $\mathbf{n}\times\mathbf{n}'$ Its equation is therefore

$$\mathbf{r} = l\mathbf{n} + l'\mathbf{n}' + t\mathbf{n}\times\mathbf{n}',$$

where t is a variable number, and l, l' have the values found above.

32. Distance of a point from a line. To find the perpendicular distance of a point P' from the straight line

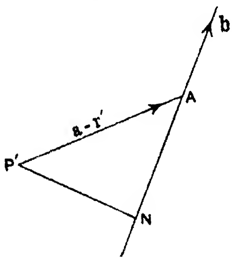

FIG. 33.

$$\mathbf{r} = \mathbf{a} + t\mathbf{b},$$

where $\mathbf{b}$ is a *unit* vector.

Let P', A be the points whose position vectors are $\mathbf{r}'$, $\mathbf{a}$ respectively, and $P'N$ the perpendicular from P' to the given line. Then

$$\overrightarrow{P'A} = \mathbf{a} - \mathbf{r}',$$

and
$$P'N^2 = P'A^2 - NA^2$$
$$= (\mathbf{a} - \mathbf{r}')^2 - \{\mathbf{b}\cdot(\mathbf{a} - \mathbf{r}')\}^2. \quad (1)$$

This equation gives the length p of the perpendicular $P'N$. As a vector

$$\overrightarrow{P'N} = \overrightarrow{P'A} - \overrightarrow{NA}$$
$$= (\mathbf{a} - \mathbf{r}') - \mathbf{b}\cdot(\mathbf{a} - \mathbf{r}')\mathbf{b}, \quad \ldots\ldots\ldots\ldots(2)$$

and p is the module of this vector.

Other theorems relating to the geometry of the plane and the straight line will be considered in the next chapter, after products of three vectors have been dealt with.

Geometry of the Sphere.

33. Equation of a sphere. Consider the sphere of centre C and radius a. If P is any point on the surface of the sphere, and $\mathbf{r}$, $\mathbf{c}$ are the position vectors of P, C respectively relative to an origin O, the vector $\overrightarrow{CP} = \mathbf{r} - \mathbf{c}$ has a length equal to the radius, and therefore

$$(\mathbf{r} - \mathbf{c})^2 = a^2.$$

If we put $k = c^2 - a^2$ we may write this relation

$$\mathbf{r}^2 - 2\mathbf{r}\cdot\mathbf{c} + k = 0. \qquad \text{(1)}$$

This equation is satisfied by the position vector of every point on the surface of the sphere, and by no other. It will therefore be called the equation of the sphere relative to the origin O.

The first member of (1), regarded as a function of the vector $\mathbf{r}$, may for convenience be denoted by $F(\mathbf{r})$; and the equation written briefly $F(\mathbf{r}) = 0$.

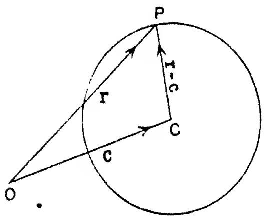

FIG. 34

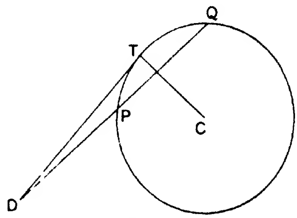

FIG. 35

Consider the points of intersection of the surface with the straight line through the point D parallel to the *unit* vector $\mathbf{b}$. The equation of this line is

$$\mathbf{r} = \mathbf{d} + t\mathbf{b}, \qquad \text{(2)}$$

where $\mathbf{d}$ is the position vector of D. The values of $\mathbf{r}$ for the points of intersection satisfy both (1) and (2). If then we eliminate $\mathbf{r}$ from these equations we find, for the values of t corresponding to the points of intersection, the quadratic equation

$$t^2 + 2\mathbf{b}\cdot(\mathbf{d} - \mathbf{c})t + (\mathbf{d}^2 - 2\mathbf{c}\cdot\mathbf{d} + k) = 0, \qquad \text{(3)}$$

the coefficient of t^2 being unity, since $\mathbf{b}$ is a unit vector. The equation has two roots, t_1, t_2 which are real if

$$\{\mathbf{b}\cdot(\mathbf{d} - \mathbf{c})\}^2 > F(\mathbf{d}),$$

the meaning of which will be apparent presently. Corresponding to these two roots there are two points of intersection, P, Q, such that $DP = t_1$ and $DQ = t_2$. The product of these roots is equal to the absolute term of (3), *i.e.*

$$DP \,.\, DQ = F(\mathbf{d}).$$

This is independent of **b**, and is therefore the same for all straight lines through D. If the points P, Q tend to coincidence at T, the straight line becomes a tangent, and we have

$$DT^2 = DP.\, DQ = F(\mathbf{d}). \quad \text{.............................(4)}$$

Thus $F(\mathbf{d})$ measures the square on the tangent from D to the surface of the sphere, and all such tangents have the same length. In particular, $F(0) = k$ is the square on the tangent from the origin. If O is within the sphere k is negative, and the tangents from O are imaginary. The tangents from D are generators of a cone, called the *tangent cone*, having its vertex at D and enveloping the sphere.

34. Equation of the tangent plane at a point. If D is a point on the surface of the sphere, $F(\mathbf{d}) = 0$, and one root of the equation (3) is zero. In order that the line (2) should touch the surface, the other root also must be zero, for a tangent line intersects the surface in two coincident points. If then both roots of (3) are zero,

$$\mathbf{b}\cdot(\mathbf{d} - \mathbf{c}) = 0,$$

showing that the tangent line is perpendicular to the radius CD. If $\mathbf{r}$ is any point on the tangent line, this condition is equivalent to

$$(\mathbf{r} - \mathbf{d})\cdot(\mathbf{d} - \mathbf{c}) = 0. \quad \text{..................................(5)}$$

This equation represents a plane through D perpendicular to CD; showing that all tangent lines through D lie on this plane, which is called the tangent plane at D. Adding the zero quantity $F(\mathbf{d})$ to the first member of (5), we may write the equation

$$\mathbf{r}\cdot\mathbf{d} - \mathbf{c}\cdot(\mathbf{r} + \mathbf{d}) + k = 0, \quad \text{................................(6)}$$

which we shall take as the standard form of the equation of the tangent plane at the point **d**.

We can now interpret the above condition for reality of the roots of (3). For $\mathbf{b}\cdot(\mathbf{d} - \mathbf{c})$ is the projection of CD along **b**, D being any point, and $F(\mathbf{d})$ is the square of the projection of CD on the tangent DT. Hence for reality of the roots the angle of

inclination of the line (2) with DC must be less than that of the tangent with DC. That is to say, the line (2) must lie within the tangent cone from D.

Since a tangent plane is perpendicular to the radius to the point of contact, the square of the perpendicular from the centre C to a tangent plane must be equal to a^2. Hence *the condition that the plane* $\mathbf{r}\cdot\mathbf{n} = q$ *should touch the sphere* (1) *is that*

$$\left(\frac{q - \mathbf{c}\cdot\mathbf{n}}{n}\right)^2 = c^2 - k. \quad \text{.............................(7)}$$

Further, if two spheres cut each other at right angles, the tangent plane to either at a point of intersection passes through the centre of the other. Hence the square on the line joining their centres is equal to the sum of the squares on their radii. If then the equations of the two spheres are

$$\left.\begin{aligned} \mathbf{r}^2 - 2\mathbf{r}\cdot\mathbf{c} + k = 0, \\ \mathbf{r}^2 - 2\mathbf{r}\cdot\mathbf{c}' + k' = 0, \end{aligned}\right\}$$

the *condition that they should cut orthogonally* is

$$\begin{aligned} (\mathbf{c} - \mathbf{c}')^2 &= a^2 + a'^2 \\ &= c^2 - k + c'^2 - k', \end{aligned}$$

that is $$2\mathbf{c}\cdot\mathbf{c}' - (k + k') = 0. \quad \text{....................................(8)}$$

35. Polar plane of a point. The tangent plane at the point $\mathbf{d}$ to the sphere whose equation is (1) is represented by (6). If this plane passes through the point $\mathbf{h}$, the vector $\mathbf{h}$ must satisfy (6); that is

$$\mathbf{h}\cdot\mathbf{d} - \mathbf{c}\cdot(\mathbf{h} + \mathbf{d}) + k = 0.$$

Further, every point such as $\mathbf{d}$, the tangent plane at which passes through $\mathbf{h}$, satisfies this relation, and therefore lies on the plane whose equation is

$$\mathbf{r}\cdot\mathbf{h} - \mathbf{c}\cdot(\mathbf{r} + \mathbf{h}) + k = 0. \quad \text{....(9)}$$

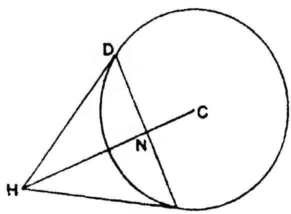

FIG. 36.

This is called the *polar plane* of the point $\mathbf{h}$. Since the terms involving $\mathbf{r}$ may be written $\mathbf{r}\cdot(\mathbf{h} - \mathbf{c})$, the plane is perpendicular to the straight line joining the point $\mathbf{h}$ to the centre of the sphere.

W. V. A.

If this straight line cuts the polar plane at N, the length of the perpendicular CN from C to the polar plane is, by Art. 30,

$$CN = \frac{-k - \mathbf{c}\cdot\mathbf{h} + \mathbf{c}\cdot(\mathbf{c}+\mathbf{h})}{\sqrt{(\mathbf{h}-\mathbf{c})^2}} = \frac{a^2}{CH},$$

where H is the point whose position vector is $\mathbf{h}$. This shows that N and H are inverse points with respect to the sphere.

The polar plane of H cuts the sphere in a circle, and the line joining H to any point on this circle is a tangent line. All such tangent lines together form the *tangent cone* from H to the sphere.

Theorem 1. *If the polar plane of the point H passes through G, the polar plane of G passes through H.* Let $\mathbf{h}$, $\mathbf{g}$ be the position vectors of the two points. Then if $\mathbf{g}$ lies on the polar plane of $\mathbf{h}$ it must satisfy (9); that is

$$\mathbf{g}\cdot\mathbf{h} - \mathbf{c}\cdot(\mathbf{g}+\mathbf{h}) + k = 0,$$

and the symmetry of this relation shows that $\mathbf{h}$ lies on the polar plane of $\mathbf{g}$.

Theorem 2. *Any straight line drawn from a point O to intersect a sphere is cut harmonically by the surface and the polar plane of O.*

Take the point O as origin, and let (1) be the equation of the surface. Then, by (9), the polar plane of O is

$$\mathbf{r}\cdot\mathbf{c} = k. \qquad \text{.................................... (10)}$$

The equation of the straight line through O parallel to the unit vector $\mathbf{b}$ is $\mathbf{r} = t\mathbf{b}$; and the value t_1 of t for the point R at which this line cuts the polar plane is given by

$$t_1\mathbf{b}\cdot\mathbf{c} = k.$$

Similarly the values t_2 and t_3 of t for the points P, Q at which the same line cuts the sphere are the roots of the equation

$$t^2 - 2t\mathbf{b}\cdot\mathbf{c} + k = 0.$$

Hence $$\frac{1}{t_2} + \frac{1}{t_3} = \frac{t_2+t_3}{t_2t_3} = \frac{2\mathbf{b}\cdot\mathbf{c}}{k} = \frac{2}{t_1}.$$

But t_1, t_2, t_3 measure the lengths of OR, OP, OQ respectively; and therefore

$$\frac{1}{OP} + \frac{1}{OQ} = \frac{2}{OR},$$

which proves the theorem.

36. Diametral plane. Consider again the points of intersection of the sphere (1) with the straight line

$$\mathbf{r} = \mathbf{d} + t\mathbf{b}, \quad \text{.................................. (2)}$$

through the point $\mathbf{d}$ parallel to $\mathbf{b}$. The roots of the equation (3) will be equal and opposite if

$$\mathbf{b}\cdot(\mathbf{d} - \mathbf{c}) = 0,$$

and $\mathbf{d}$ will then be the mid point of the chord joining the points of intersection. This equation shows that all points such as $\mathbf{d}$, bisecting chords parallel to $\mathbf{b}$, lie on the plane whose equation is

$$\mathbf{b}\cdot(\mathbf{r} - \mathbf{c}) = 0.$$

This plane, which passes through the centre of the sphere and is perpendicular to the chords it bisects, is called the *diametral plane* of the sphere for chords parallel to $\mathbf{b}$.

Further, if $\mathbf{r}$ is any point on the straight line (2), the above condition for equal and opposite roots of the quadratic in t shows that

$$(\mathbf{r} - \mathbf{d})\cdot(\mathbf{d} - \mathbf{c}) = 0,$$

which is the equation of a plane through the point $\mathbf{d}$ perpendicular to the line joining $\mathbf{d}$ to the centre of the sphere. Thus all chords of the sphere, which are bisected by the point D, lie in the plane through D perpendicular to CD.

37. Radical plane of two spheres. If the tangents drawn from the point $\mathbf{d}$ to the two spheres

$$\left.\begin{aligned} F(\mathbf{r}) &\equiv \mathbf{r}^2 - 2\mathbf{r}\cdot\mathbf{c} + k = 0, \\ F'(\mathbf{r}) &\equiv \mathbf{r}^2 - 2\mathbf{r}\cdot\mathbf{c}' + k' = 0 \end{aligned}\right\}$$

are equal in length, $F(\mathbf{d}) = F'(\mathbf{d})$; that is

$$2\mathbf{d}\cdot(\mathbf{c} - \mathbf{c}') = k - k'.$$

Thus the point $\mathbf{d}$ lies on the plane whose equation is

$$2\mathbf{r}\cdot(\mathbf{c} - \mathbf{c}') = k - k'. \quad \text{............................(11)}$$

This is called the *radical plane* of the two spheres. It is perpendicular to the straight line joining their centres; and the tangents drawn from any point on it to the two spheres are equal in length. If the spheres intersect, all the points of intersection lie on the radical plane. For any value of $\mathbf{r}$ satisfying the equations of both spheres also satisfies $F(\mathbf{r}) - F'(\mathbf{r}) = 0$, which is the equation of the radical plane.

We shall now consider a *system of spheres with a common radical plane.* Let O be the origin, and A, B the two points $\mathbf{a}$, $-\mathbf{a}$ respectively on the surface of the sphere whose equation is $\mathbf{r}^2=\mathbf{a}^2$. Consider the spheres whose centres lie on the straight line AB, and which cut the above sphere orthogonally. The

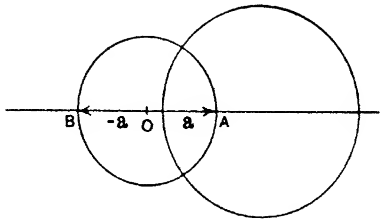

FIG. 37.

centre of any one of these is a point $m\mathbf{a}$, where m is a real number, positive or negative, and the equation of the sphere is then

$$\mathbf{r}^2-2m\mathbf{r}\cdot\mathbf{a}+k=0.$$

But by (8), if this cuts the sphere $\mathbf{r}^2=a^2$ orthogonally, we must have $k=a^2$. Hence the required system of spheres is represented by the equation

$$\mathbf{r}^2-2m\mathbf{r}\cdot\mathbf{a}+a^2=0, \quad\quad (12)$$

m denoting a real number, positive or negative. The radius of a sphere of the system depends on the value of m, the square of the radius being

$$c^2-k=m^2a^2-a^2=(m^2-1)a^2.$$

Thus the spheres increase in size as their centres get further from the origin. Since the square of the radius must be positive, the value of m cannot lie between ± 1. For these limiting values of m the radius of the sphere is zero, the centres of the vanishing spheres being A and B. These points are therefore called the *limiting points* of the system of spheres.

The radical plane of the two spheres corresponding to the values m_1 and m_2 is, in virtue of (11),

$$2\mathbf{r}\cdot(m_1\mathbf{a}-m_2\mathbf{a})=0,$$

that is $$\mathbf{r}\cdot\mathbf{a}=0,$$

which is the plane through the origin perpendicular to AB, and is the same for all pairs of spheres. Thus the system of spheres

has a common radical plane. If C_1 and C_2 are the centres of the spheres for the values m_1 and m_2, and R_1 and R_2 their radii,

$$\begin{aligned} R_1^2 - R_2^2 &= (m_1^2 - 1)a^2 - (m_2^2 - 1)a^2 \\ &= m_1^2a^2 - m_2^2a^2 \\ &= OC_1^2 - OC_2^2, \end{aligned}$$

which is a relation showing how the radical plane divides the join of the centres of two spheres.

The limiting points A, B are inverse points with respect to any sphere of the system. For if C is the centre $m\mathbf{a}$, and R the radius of any sphere,

$$AC \,.\, BC = (m\mathbf{a} - \mathbf{a})\cdot(m\mathbf{a} + \mathbf{a}) = (m^2 - 1)a^2 = R^2,$$

which proves the statement. It follows that the polar plane of either limiting point, with respect to any sphere of the system, passes through the other limiting point, and is perpendicular to the line joining them.

38. If $F(\mathbf{r}) = 0$ and $F'(\mathbf{r}) = 0$ are the equations of two spheres, then

$$F(\mathbf{r}) - \lambda F'(\mathbf{r}) = 0 \qquad (13)$$

also represents a sphere, λ being an arbitrary number. For, on division by $(1 - \lambda)$ this equation takes the form

$$\mathbf{r}^2 - 2\mathbf{r}\cdot\left(\frac{\mathbf{c} - \lambda\mathbf{c}'}{1 - \lambda}\right) + \left(\frac{k - \lambda k'}{1 - \lambda}\right) = 0, \qquad (13')$$

which represents a sphere whose centre is the point $(\mathbf{c} - \lambda\mathbf{c}')/(1 - \lambda)$ on the straight line through the centres of the two given spheres. It is easily shown that any two spheres given by (13′) for different values of λ have a radical plane

$$2\mathbf{r}\cdot(\mathbf{c} - \mathbf{c}') = k - k', \qquad (14)$$

which is the same for any two spheres, and is the radical plane of the two given spheres. Hence the equation (13), for real values of λ, represents a system of spheres with a common radical plane, perpendicular to the line of centres.

If the two spheres $F(\mathbf{r}) = 0$ and $F'(\mathbf{r}) = 0$ intersect, the radical plane and all spheres of the system (13) pass through their circle of intersection. For any value of $\mathbf{r}$ satisfying the equations of both spheres clearly satisfies both (13) and (14).

Application to Mechanics.

39. Work done by a force. A force acting on a particle does work when the particle is displaced in a direction which is not perpendicular to the force. The work done is a scalar quantity jointly proportional to the force and the resolved part of the displacement in the direction of the force. We choose the unit quantity of work as that done when a particle, acted on by unit force, is displaced unit distance in the direction of the force. Hence, if $\mathbf{F}$, $\mathbf{d}$ are vectors representing the force and the displacement respectively, inclined at an angle θ, the measure of the work done is

$$Fd\cos\theta = \mathbf{F}\cdot\mathbf{d}.$$

The work done is zero only when $\mathbf{d}$ is perpendicular to $\mathbf{F}$.

Suppose next that the particle is acted on by several forces $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n$. Then during a displacement $\mathbf{d}$ of the particle the separate forces do quantities of work $\mathbf{F}_1\cdot\mathbf{d}, \mathbf{F}_2\cdot\mathbf{d}, \ldots, \mathbf{F}_n\cdot\mathbf{d}$. The total work done is

$$\sum_1^n \mathbf{F}\cdot\mathbf{d} = \mathbf{d}\cdot\Sigma\mathbf{F} = \mathbf{d}\cdot\mathbf{R},$$

and is therefore the same as if the system of forces were replaced by its resultant $\mathbf{R}$.

Note. *A force represented by the vector* $\mathbf{F}$ *may be conveniently referred to as a force* $\mathbf{F}$. *No misunderstanding is possible, for our Clarendon symbols always denote length-vectors. Similarly we may speak of a displacement* $\mathbf{d}$, *or a velocity* $\mathbf{v}$, *as we have already done of a point* $\mathbf{r}$.

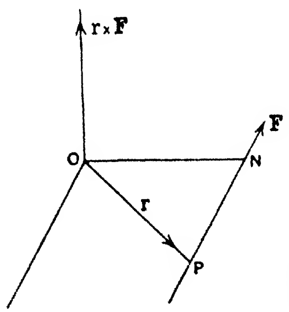

FIG. 38.

40. Vector moment or torque of a force. The vector moment (or briefly the moment) of a force $\mathbf{F}$ about a point O is a vector quantity related to an axis through O perpendicular to the plane containing O and the line of action of the force. Its magnitude is jointly proportional to the force and the perpendicular distance ON upon its line of action. The moment or *torque*

of the force about O may therefore be represented by a vector perpendicular to the plane of $\mathbf{r}$ and $\mathbf{F}$, where $\mathbf{r}$ is the position vector relative to O of any point P on the line of action of the force. And with the choice of unit moment as that of unit force localised in a line unit distance from O, the vector representing the moment of $\mathbf{F}$ about O is $\mathbf{r}\times\mathbf{F}$. For the direction of this vector is positive relative to the rotation about the axis through O in the sense indicated by the direction of the force; and the module of the vector is

$$F.\,OP.\sin O\hat{P}N = F.\,ON$$

as required.

If there are several forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3\ldots$ acting through the same point P they have a resultant $\mathbf{R}=\Sigma\mathbf{F}$. The moment of this resultant about O is

$$\begin{aligned}\mathbf{r}\times\mathbf{R} &= \mathbf{r}\times(\mathbf{F}_1+\mathbf{F}_2+\ldots)\\ &= \mathbf{r}\times\mathbf{F}_1+\mathbf{r}\times\mathbf{F}_2+\ldots,\end{aligned}$$

and is therefore equal to the vector sum of the moments of the separate forces. Thus, *if the system of forces through P is replaced by its resultant, the moment about any point remains unchanged.*

Express the vectors $\mathbf{F}$, $\mathbf{r}$ in terms of their rectangular components, as

$$\begin{aligned}\mathbf{F} &= X\mathbf{i}+Y\mathbf{j}+Z\mathbf{k},\\ \mathbf{r} &= x\mathbf{i}+y\mathbf{j}+z\mathbf{k}.\end{aligned}$$

Then the moment of the force $\mathbf{F}$ about O is

$$\mathbf{M}=(yZ-zY)\mathbf{i}+(zX-xZ)\mathbf{j}+(xY-yX)\mathbf{k}.$$

In this expression the coefficients of $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are the ordinary scalar moments of the force about the coordinate axes. And as the system of rectangular axes through O may be chosen so that one of them has any assigned direction, it follows that *the ordinary moment of a force $\mathbf{F}$ about any straight line through O is the resolved part along this line of the vector moment of $\mathbf{F}$ about O.* With the above notation the scalar moments about the coordinate axes are $\mathbf{M}\cdot\mathbf{i}$, $\mathbf{M}\cdot\mathbf{j}$ and $\mathbf{M}\cdot\mathbf{k}$ respectively.

If there are several concurrent forces it follows from the above that the scalar moment of the resultant about any axis through O is equal to the sum of the scalar moments of the several forces about that axis.

In agreement with the above we adopt the following

Definition. *The moment about the origin of the vector* $\mathbf{v}$ *localised in a line through the point* $\mathbf{r}$ *is the vector* $\mathbf{r}\times\mathbf{v}$.

41. Angular velocity of a rigid body about a fixed axis. Consider the motion of a rigid body rotating about a fixed axis ON at the rate of ω radians per second. It will be shown in Art. 87 that, if one point O of the body is fixed, the instantaneous motion of the body is one of rotation about such an axis through O, every point on the axis being instantaneously at rest. For the present we take the rotation as given about the axis ON. The angular velocity of the body is uniquely specified by a vector $\mathbf{A}$ whose module is ω and whose direction is parallel to the axis, and in the positive sense relative to the rotation.

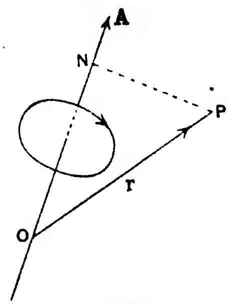

FIG. 39.

Let O be any point on the fixed axis, P a point fixed in the body, $\mathbf{r}$ the position vector of P relative to O, and PN perpendicular to the axis of rotation. Then the particle at P is moving in a circular path, with centre N and radius $p=PN$. Its velocity is therefore perpendicular to the plane OPN and of magnitude $p\omega = r\omega \sin P\hat{O}N$. Such a velocity is represented by the vector $\mathbf{A}\times\mathbf{r}$, the sense of this vector being the same as that of the velocity. In other words, the velocity of the particle at P is

$$\mathbf{v}=\mathbf{A}\times\mathbf{r}.$$

Examples.

(1) *A rigid body is spinning with an angular velocity of 4 radians per second about an axis parallel to* $3\mathbf{j}-\mathbf{k}$ *passing through the point* $\mathbf{i}+3\mathbf{j}-\mathbf{k}$. *Find the velocity of the particle at the point* $4\mathbf{i}-2\mathbf{j}+\mathbf{k}$.

Relative to the given point on the axis of rotation the position vector of the particle is

$$\mathbf{r}=3\mathbf{i}-5\mathbf{j}+2\mathbf{k}.$$

The vector specifying the angular velocity is

$$\mathbf{A}=\frac{4(3\mathbf{j}-\mathbf{k})}{|3\mathbf{j}-\mathbf{k}|}=\frac{4}{\sqrt{10}}(3\mathbf{j}-\mathbf{k}).$$

Hence the velocity of the particle is given by

$$\mathbf{A}\times\mathbf{r} = \frac{4}{\sqrt{10}}(3\mathbf{j}-\mathbf{k})\times(3\mathbf{i}-5\mathbf{j}+2\mathbf{k})$$

$$= \frac{4}{\sqrt{10}}(\mathbf{i}-3\mathbf{j}-9\mathbf{k}).$$

This represents a speed of $4\sqrt{\frac{91}{10}}=12$ approximately, in the direction of the vector $\mathbf{A}\times\mathbf{r}$.

(2) *Laws of Reflection and Refraction of Light.* Let $\mathbf{n}$, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be *unit* vectors, the first normal to the surface of separation of two media, and the others in the directions of the incident, reflected and refracted rays; the laws of reflection and refraction are specified by

$$\mathbf{a}\times\mathbf{n}=\mathbf{b}\times\mathbf{n},$$

and

$$\mu\mathbf{a}\times\mathbf{n}=\mu'\mathbf{c}\times\mathbf{n}$$

respectively, where μ, μ' are the indices of refraction for the two media.

For the first equation makes the incident and reflected rays coplanar with the normal, as well as the angles of incidence and reflection equal. The second makes the incident and refracted rays coplanar with the normal, and also gives

$$\mu \sin i = \mu' \sin r,$$

where i and r are the angles of incidence and refraction respectively.

EXERCISES ON CHAPTER III.

Give *vectorial* solutions of the following:

1. The perpendiculars let fall from the vertices of a triangle to the opposite sides are concurrent.

2. The perpendicular bisectors of the sides of a triangle are concurrent.

3. The vector area of the triangle whose vertices are the points $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ is $\frac{1}{2}(\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}+\mathbf{a}\times\mathbf{b})$.

4. The area of the triangle formed by joining the mid point of one of the non-parallel sides of a trapezium to the extremities of the opposite side is half that of the trapezium.

5. What is the unit vector perpendicular to each of the vectors $2\mathbf{i}-\mathbf{j}+\mathbf{k}$ and $3\mathbf{i}+4\mathbf{j}-\mathbf{k}$? Calculate the sine of the angle between these two vectors.

6. Find the torque about the point $\mathbf{i}+2\mathbf{j}-\mathbf{k}$ of a force represented by $3\mathbf{i}+\mathbf{k}$ acting through the point $2\mathbf{i}-\mathbf{j}+3\mathbf{k}$.

7. Show that twice the vector area of a closed plane polygon is *equivectorial* * with the sum of the torques about *any* point of forces represented by the sides of the polygon taken in order.

8. Find the equation of the straight line through the point $\mathbf{d}$, equally inclined to the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$.

9. If a straight line is equally inclined to three coplanar straight lines, it is perpendicular to their plane.

10. If a point is equidistant from the vertices of a right-angled triangle, its join to the mid point of the hypotenuse is perpendicular to the plane of the triangle.

11. From an external point O a perpendicular ON is drawn to a plane, and another OM to a straight line PQ in the plane. Prove that MN is perpendicular to PQ.

12. Find the two vectors which are equally inclined to $\mathbf{k}$, and are perpendicular to each other and to $\mathbf{i}+\mathbf{j}+\mathbf{k}$.

13. The sum of the squares on the edges of any tetrahedron is equal to four times the sum of the squares on the joins of the mid points of opposite edges.

14. Prove that in any parallelogram the sum of the squares on the diagonals is twice the sum of the squares on two adjacent sides; the difference of the squares on the diagonals is four times the rectangle contained by either of these sides and the projection of the other upon it; and the difference of the squares on two adjacent sides is equal to the rectangle contained by either diagonal and the projection of the other upon it.

15. Show that, in a regular tetrahedron, the perpendiculars from the vertices to the opposite faces meet those faces at their centroids. Find the angle between two faces, and the angle between a face and an edge which cuts it.

16. Find the equation of the plane through the point $2\mathbf{i}+3\mathbf{j}-\mathbf{k}$ perpendicular to the vector $3\mathbf{i}-4\mathbf{j}+7\mathbf{k}$.

17. Find the equation of the plane passing through the origin, and the line of intersection of $\mathbf{r}\cdot\mathbf{a}=p$ and $\mathbf{r}\cdot\mathbf{b}=q$.

18. Show that the points $\mathbf{i}-\mathbf{j}+3\mathbf{k}$ and $3(\mathbf{i}+\mathbf{j}+\mathbf{k})$ are equidistant from the plane $\mathbf{r}\cdot(5\mathbf{i}+2\mathbf{j}-7\mathbf{k})+9=0$, and on opposite sides of it.

19. Find the equation of the line of intersection of $\mathbf{r}\cdot(3\mathbf{i}-\mathbf{j}+\mathbf{k})=1$, and $\mathbf{r}\cdot(\mathbf{i}+4\mathbf{j}-2\mathbf{k})=2$.

20. Determine the plane through the point $\mathbf{i}+2\mathbf{j}-\mathbf{k}$, which is perpendicular to the line of intersection of the planes in the previous exercise.

* *i.e.* specified by the same vector.

21. Find the perpendicular distance of a corner of a unit cube from a diagonal not passing through it.

22. Show that the sum of the reciprocals of the squares of the intercepts on rectangular axes made by a fixed plane is the same for all systems of rectangular axes with a given origin.

23. Find the coordinates of the centre of the sphere inscribed in the tetrahedron bounded by the planes

$$\mathbf{r}\cdot\mathbf{i}=0,\quad \mathbf{r}\cdot\mathbf{j}=0,\quad \mathbf{r}\cdot\mathbf{k}=0,\quad \text{and}\quad \mathbf{r}\cdot(\mathbf{i}+\mathbf{j}+\mathbf{k})=a.$$

24. Show that the planes $\mathbf{r}\cdot(2\mathbf{i}+5\mathbf{j}+3\mathbf{k})=0$, $\mathbf{r}\cdot(\mathbf{i}-\mathbf{j}+4\mathbf{k})=2$, and $\mathbf{r}\cdot(7\mathbf{j}-5\mathbf{k})+4=0$ have a common line of intersection.

25. The line of intersection of $\mathbf{r}\cdot(\mathbf{i}+2\mathbf{j}+3\mathbf{k})=0$ and $\mathbf{r}\cdot(3\mathbf{i}+2\mathbf{j}+\mathbf{k})=0$ is equally inclined to $\mathbf{i}$ and $\mathbf{k}$, and makes an angle $\frac{1}{2}\sec^{-1}3$ with $\mathbf{j}$.

26. Find the locus of a point which moves so that the difference of the squares of its distances from two given points is constant.

27. Find the locus of a point about which two given coplanar forces have equal moments.

28. Three forces P, $2P$, $3P$ act along the sides AB, BC, CA respectively of an equilateral triangle ABC. Find their resultant, and the point in which its line of action cuts BC.

29. A particle, acted on by constant forces $4\mathbf{i}+\mathbf{j}-3\mathbf{k}$ and $3\mathbf{i}+\mathbf{j}-\mathbf{k}$, is displaced from the point $\mathbf{i}+2\mathbf{j}+3\mathbf{k}$ to the point $5\mathbf{i}+4\mathbf{j}+\mathbf{k}$. Find the total work done by the forces.

30. Show that any diameter of a sphere subtends a right angle at a point on the surface.

31. The equation of a sphere which has the points $\mathbf{g}$, $\mathbf{h}$ as the extremities of a diameter is $(\mathbf{r}-\mathbf{g})\cdot(\mathbf{r}-\mathbf{h})=0$.

32. The locus of a point, the sum of the squares on whose distances from n given points is constant, is a sphere.

33. The locus of a point which moves so that its distances from two fixed points are in a constant ratio $n:1$ is a sphere. Also show that all such spheres, for different values of n, have a common radical plane.

34. From a point O a straight line is drawn to meet a fixed sphere in P. In OP a point Q is taken so that $OP:OQ$ is a fixed ratio. Find the locus of Q.

35. Find the inverse of a sphere with respect to a point (i) outside the sphere, (ii) on its surface.

36. The distances of two points from the centre of a given sphere are proportional to the distances of the points each from the polar plane of the other.

37. The sphere which cuts $F(\mathbf{r})=0$ and $F'(\mathbf{r})=0$ orthogonally also cuts $F(\mathbf{r})-\lambda F'(\mathbf{r})=0$ orthogonally.

38. If, from any point on the surface of a sphere, straight lines are drawn to the extremities of any diameter of a concentric sphere, the sum of the squares on these lines is constant.

39. From the relations

$$\begin{cases} \mathbf{r}'=\mathbf{r}+\left[\dfrac{\gamma-1}{\mathbf{v}^2}\,\mathbf{v}\cdot\mathbf{r}-\gamma t\right]\mathbf{v}, \\ t'=\gamma\left[t-\dfrac{\mathbf{v}\cdot\mathbf{r}}{c^2}\right], \end{cases}$$

where $\gamma=c/\sqrt{c^2-\mathbf{v}^2}$, prove the reciprocal relations

$$\begin{cases} \mathbf{r}=\mathbf{r}'+\left[\dfrac{\gamma-1}{\mathbf{v}^2}\,\mathbf{v}\cdot\mathbf{r}'+\gamma t'\right]\mathbf{v}, \\ t=\gamma\left[t'+\dfrac{\mathbf{v}\cdot\mathbf{r}'}{c^2}\right]. \end{cases}$$

(These formulae occur in the theory of Relativity.)

40. Prove that, in a linear relation connecting the position vectors of any number of coplanar points, the algebraic sum of the coefficients is zero (the origin not being in the plane of the points).

CHAPTER IV.

PRODUCTS OF THREE OR FOUR VECTORS NON-INTERSECTING STRAIGHT LINES.

Products of Three Vectors.

42. Since the cross product $\mathbf{b}\times\mathbf{c}$ of the vectors $\mathbf{b}$, $\mathbf{c}$ is itself a vector, we may form with it and a third vector $\mathbf{a}$ both the scalar product $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ and the vector product $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$. The former is a number and the latter a vector. Such triple products are of frequent occurrence, and we shall find it useful to examine their properties.

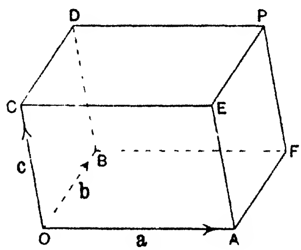

FIG. 40

43. Scalar triple product, $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$. Consider the parallelepiped whose concurrent edges OA, OB, OC have the lengths and directions of the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ respectively. Then the vector $\mathbf{b}\times\mathbf{c}$, which we may denote by $\mathbf{n}$, is perpendicular to the face $OBDC$, and its module n is the measure of the area of that face. If θ is the angle between the directions of $\mathbf{n}$ and $\mathbf{a}$, the triple product

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = an\cos\theta = V,$$

where V is the measure of the volume of the parallelepiped. The triple product is positive if θ is acute, that is if $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ form a right-handed system of vectors.

The same reasoning shows that each of the products $\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})$ and $\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$ is equal to V, and therefore to the original product. The cyclic order $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ is maintained in each of these. If,

however, that order is changed, the sign of the product is changed. For since $\mathbf{b}\times\mathbf{c} = -\mathbf{c}\times\mathbf{b}$ it follows that

$$\begin{aligned} V &= \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = (\mathbf{b}\times\mathbf{c})\cdot\mathbf{a} = -(\mathbf{c}\times\mathbf{b})\cdot\mathbf{a} = -\mathbf{a}\cdot(\mathbf{c}\times\mathbf{b}) \\ &= \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = (\mathbf{c}\times\mathbf{a})\cdot\mathbf{b} = -(\mathbf{a}\times\mathbf{c})\cdot\mathbf{b} = -\mathbf{b}\cdot(\mathbf{a}\times\mathbf{c}) \\ &= \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}) = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = -(\mathbf{b}\times\mathbf{a})\cdot\mathbf{c} = -\mathbf{c}\cdot(\mathbf{b}\times\mathbf{a}). \end{aligned}$$

Thus the value of the product depends on the cyclic order of the factors, but is *independent of the position of the dot and cross. These may be interchanged at pleasure.* It is usual to denote the above product by $[\mathbf{abc}]$, which indicates the three factors and their cyclic order. Then

$$[\mathbf{abc}] = -[\mathbf{acb}].$$

If the three vectors are coplanar their scalar triple product is zero. For $\mathbf{b}\times\mathbf{c}$ is then perpendicular to $\mathbf{a}$, and their scalar product vanishes. Thus the vanishing of $[\mathbf{abc}]$ is the *condition that the vectors should be coplanar.* If two of the vectors are parallel this condition is satisfied. In particular, if two of them are equal the product is zero.

Expressing the vectors in terms of their rectangular components, $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, and so on, we have

$$\begin{aligned} [\mathbf{abc}] &= a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1) \\ &= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \end{aligned}$$

This is the well known expression for the volume of a parallelepiped with one corner at the origin. More generally, if in terms of three non-coplanar vectors $\mathbf{l}$, $\mathbf{m}$, $\mathbf{n}$ we write

$$\mathbf{a} = a_1\mathbf{l} + a_2\mathbf{m} + a_3\mathbf{n},$$

and so on, it is easily shown that

$$[\mathbf{abc}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} [\mathbf{lmn}].$$

The product $[\mathbf{ijk}]$, of three rectangular unit vectors, is obviously equal to unity.

44. Vector triple product. Consider next the cross product of $\mathbf{a}$ and $\mathbf{b}\times\mathbf{c}$, viz.

$$\mathbf{P} = \mathbf{a}\times(\mathbf{b}\times\mathbf{c}).$$

This is a vector perpendicular to both **a** and **b**×**c**. But **b**×**c** is normal to the plane of **b** and **c**, so that **P** must lie in this plane. It is therefore expressible in terms of **b** and **c** in the form

$$\mathbf{P} = l\mathbf{b} + m\mathbf{c}.$$

To find the actual expression for **P** consider unit vectors **j** and **k**, the first parallel to **b** and the second perpendicular to it in the plane **b**, **c**. Then we may put

$$\mathbf{b} = b\mathbf{j},$$
$$\mathbf{c} = c_2\mathbf{j} + c_3\mathbf{k}.$$

In terms of **j**, **k** and the other unit vector **i** of the right-handed system, the remaining vector **a** may be written

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}.$$

Then $\mathbf{b}\times\mathbf{c} = bc_3\mathbf{i}$, and the triple product

$$\begin{aligned}\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) &= a_3bc_3\mathbf{j} - a_2bc_3\mathbf{k}\\ &= (a_2c_2 + a_3c_3)b\mathbf{j} - a_2b(c_2\mathbf{j} + c_3\mathbf{k})\\ &= \mathbf{a}\cdot\mathbf{c}\mathbf{b} - \mathbf{a}\cdot\mathbf{b}\mathbf{c}. \qquad\qquad (1)\end{aligned}$$

This is the required expression for **P** in terms of **b** and **c**.

Similarly the triple product

$$\begin{aligned}(\mathbf{b}\times\mathbf{c})\times\mathbf{a} &= -\mathbf{a}\times(\mathbf{b}\times\mathbf{c})\\ &= \mathbf{a}\cdot\mathbf{b}\mathbf{c} - \mathbf{a}\cdot\mathbf{c}\mathbf{b}. \qquad\qquad (2)\end{aligned}$$

It will be noticed that the expansions (1) and (2) are both written down by the same rule. Each scalar product involves the factor outside the bracket; and the first is the scalar product of the extremes.

In a vector triple product the position of the brackets cannot be changed without altering the value of the product. For $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$ is a vector expressible in terms of **a** and **b**; $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ is one expressible in terms of **b** and **c**. The products in general therefore represent different vectors.

If a vector **r** is resolved into two components in the plane of **a** and **r**, one parallel to **a** and the other perpendicular to it, the former component is $\frac{\mathbf{a}\cdot\mathbf{r}}{a^2}\mathbf{a}$, and therefore the latter

$$\mathbf{r} - \frac{\mathbf{a}\cdot\mathbf{r}}{a^2}\mathbf{a} = \frac{\mathbf{a}\cdot\mathbf{a}\mathbf{r} - \mathbf{a}\cdot\mathbf{r}\mathbf{a}}{a^2} = \frac{\mathbf{a}\times(\mathbf{r}\times\mathbf{a})}{a^2}.$$

Products of Four Vectors.

45. A scalar product of four vectors. The products already considered are usually sufficient for practical applications. But we occasionally meet with products of four vectors of the following types.

Consider the scalar product of $\mathbf{a}\times\mathbf{b}$ and $\mathbf{c}\times\mathbf{d}$. This is a number easily expressible in terms of the scalar products of the individual vectors. For, in virtue of the fact that in a scalar triple product the dot and cross may be interchanged, we may write

$$\begin{aligned}(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) &= \mathbf{a}\cdot\mathbf{b}\times(\mathbf{c}\times\mathbf{d})\\ &= \mathbf{a}\cdot(\mathbf{b}\cdot\mathbf{d}\mathbf{c} - \mathbf{b}\cdot\mathbf{c}\mathbf{d})\\ &= \mathbf{a}\cdot\mathbf{c}\,\mathbf{b}\cdot\mathbf{d} - \mathbf{a}\cdot\mathbf{d}\,\mathbf{b}\cdot\mathbf{c}.\end{aligned}$$

Writing this result in the form of a determinant, we have

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = \begin{vmatrix}\mathbf{a}\cdot\mathbf{c} & \mathbf{a}\cdot\mathbf{d}\\ \mathbf{b}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{d}\end{vmatrix}.$$

46. A vector product of four vectors.* Consider next the vector product of $\mathbf{a}\times\mathbf{b}$ and $\mathbf{c}\times\mathbf{d}$. This is a vector at right angles to $\mathbf{a}\times\mathbf{b}$, and therefore coplanar with $\mathbf{a}$ and $\mathbf{b}$. Similarly it is coplanar with $\mathbf{c}$ and $\mathbf{d}$. It must therefore be parallel to the line of intersection of a plane parallel to $\mathbf{a}$ and $\mathbf{b}$ with another parallel to $\mathbf{c}$ and $\mathbf{d}$.

To express the product $(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})$ in terms of $\mathbf{a}$ and $\mathbf{b}$, regard it as the vector triple product of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{m}$, where $\mathbf{m}=\mathbf{c}\times\mathbf{d}$. Then

$$\begin{aligned}(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}) &= (\mathbf{a}\times\mathbf{b})\times\mathbf{m} = \mathbf{a}\cdot\mathbf{m}\mathbf{b} - \mathbf{b}\cdot\mathbf{m}\mathbf{a}\\ &= [\mathbf{acd}]\mathbf{b} - [\mathbf{bcd}]\mathbf{a}. \quad\ldots\ldots(1)\end{aligned}$$

Similarly, regarding it as the vector triple product of $\mathbf{n}$, $\mathbf{c}$ and $\mathbf{d}$, where $\mathbf{n}=\mathbf{a}\times\mathbf{b}$, we may write it

$$\begin{aligned}(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}) &= \mathbf{n}\times(\mathbf{c}\times\mathbf{d}) = \mathbf{n}\cdot\mathbf{d}\mathbf{c} - \mathbf{n}\cdot\mathbf{c}\mathbf{d}\\ &= [\mathbf{abd}]\mathbf{c} - [\mathbf{abc}]\mathbf{d}. \quad\ldots\ldots(2)\end{aligned}$$

Equating these two expressions we have a relation between the four vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, viz.

$$[\mathbf{bcd}]\mathbf{a} - [\mathbf{acd}]\mathbf{b} + [\mathbf{abd}]\mathbf{c} - [\mathbf{abc}]\mathbf{d} = 0. \quad\ldots\ldots(3)$$

* The bracket notation for the above products of three and four vectors is

$$(\mathbf{a}|\mathbf{bc}|) = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})\,; \quad |\mathbf{a}|\mathbf{bc}|| = \mathbf{a}\times(\mathbf{b}\times\mathbf{c})\,;$$

$$(|\mathbf{ab}||\mathbf{cd}|) = (\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) \quad\text{and}\quad ||\mathbf{ab}||\mathbf{cd}|| = (\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}).$$

Writing $\mathbf{r}$ instead of $\mathbf{d}$, we may express any vector $\mathbf{r}$ in terms of three other vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ in the form

$$\mathbf{r} = \frac{[\mathbf{rbc}]\mathbf{a} + [\mathbf{rca}]\mathbf{b} + [\mathbf{rab}]\mathbf{c}}{[\mathbf{abc}]}, \qquad\qquad (4)$$

which is valid except when the denominator $[\mathbf{abc}]$ vanishes, that is except when $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are coplanar.

47. Reciprocal system of vectors to a, b, c. In terms of the vectors $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$ defined by the equations

$$\mathbf{a}' = \frac{\mathbf{b}\times\mathbf{c}}{[\mathbf{abc}]}; \quad \mathbf{b}' = \frac{\mathbf{c}\times\mathbf{a}}{[\mathbf{abc}]}; \quad \mathbf{c}' = \frac{\mathbf{a}\times\mathbf{b}}{[\mathbf{abc}]}$$

we may write the relation (4) above,

$$\mathbf{r} = \mathbf{r}\cdot\mathbf{a}'\mathbf{a} + \mathbf{r}\cdot\mathbf{b}'\mathbf{b} + \mathbf{r}\cdot\mathbf{c}'\mathbf{c}.$$

The vectors $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$ are called the *reciprocal system* to $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, which are assumed non-coplanar so that the denominator $[\mathbf{abc}]$ does not vanish. The reason for the name lies in the obvious relations

$$\mathbf{a}\cdot\mathbf{a}' = \mathbf{b}\cdot\mathbf{b}' = \mathbf{c}\cdot\mathbf{c}' = 1.$$

They also possess the property that the scalar product of any other pair of vectors, one from each system, is zero. For instance,

$$\mathbf{a}\cdot\mathbf{b}' = [\mathbf{aca}]/[\mathbf{abc}] = 0,$$

since, in the numerator, two factors of the scalar triple product are equal. Similarly $\mathbf{c}\cdot\mathbf{b}' = 0$.

The symmetry of the above relations shows that if $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$ is the reciprocal system to $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, then $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ is the reciprocal system to $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$. Hence any vector $\mathbf{r}$ may be expressed in terms of $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$ in the form

$$\mathbf{r} = \mathbf{r}\cdot\mathbf{a}\mathbf{a}' + \mathbf{r}\cdot\mathbf{b}\mathbf{b}' + \mathbf{r}\cdot\mathbf{c}\mathbf{c}'.$$

The system of unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ is easily seen to be its own reciprocal; and we have already proved for this case the relation

$$\mathbf{r} = \mathbf{r}\cdot\mathbf{i}\mathbf{i} + \mathbf{r}\cdot\mathbf{j}\mathbf{j} + \mathbf{r}\cdot\mathbf{k}\mathbf{k}.$$

The scalar triple product $[\mathbf{abc}]$, formed from three non-coplanar vectors, is the reciprocal of the product $[\mathbf{a}'\mathbf{b}'\mathbf{c}']$ formed from the reciprocal system. For

$$[\mathbf{a}'\mathbf{b}'\mathbf{c}'] = \left[\frac{\mathbf{b}\times\mathbf{c}}{[\mathbf{abc}]}\ \frac{\mathbf{c}\times\mathbf{a}}{[\mathbf{abc}]}\ \frac{\mathbf{a}\times\mathbf{b}}{[\mathbf{abc}]}\right] = \frac{[\mathbf{b}\times\mathbf{c},\ \mathbf{c}\times\mathbf{a},\ \mathbf{a}\times\mathbf{b}]}{[\mathbf{abc}]^3}.$$

Now, by Art. 46, $(\mathbf{b}\times\mathbf{c})\times(\mathbf{c}\times\mathbf{a}) = [\mathbf{abc}]\mathbf{c}$, and therefore the numerator of the above expression is equal to $[\mathbf{abc}]^2$. Hence the result.

Further Geometry of the Plane and Straight Line.

48. Planes satisfying various conditions. We shall now see how the triple products considered above may be used in forming the vector equation of a plane subject to certain conditions. Let us examine the following typical cases :

(i) *Plane through three given points A, B, C.* Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be the position vectors of the points relative to an assigned origin O, and $\mathbf{r}$ that of a variable point P on the plane. Since P, A, B, C all lie on the plane, the vectors $\mathbf{r}-\mathbf{a}$, $\mathbf{a}-\mathbf{b}$, $\mathbf{b}-\mathbf{c}$ are coplanar, and their scalar triple product is zero. Hence

$$(\mathbf{r}-\mathbf{a})\cdot(\mathbf{a}-\mathbf{b})\times(\mathbf{b}-\mathbf{c})=0.$$

If we expand this, and neglect the triple products in which any vector occurs twice, the equation becomes

$$\mathbf{r}\cdot(\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}+\mathbf{a}\times\mathbf{b})=[\mathbf{abc}].$$

Thus the plane is perpendicular to the vector

$$\mathbf{n}=\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}+\mathbf{a}\times\mathbf{b},$$

which represents twice the vector area of the triangle ABC. If n is the module of $\mathbf{n}$, the length p of the perpendicular ON from the origin to the plane is

$$p=\frac{[\mathbf{abc}]}{n},$$

while $$\overrightarrow{ON}=p\hat{\mathbf{n}}=\frac{1}{n^2}[\mathbf{abc}](\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}+\mathbf{a}\times\mathbf{b}).$$

(ii) *Plane through a given point parallel to two given straight lines.* Let $\mathbf{a}$ be the given point, and $\mathbf{b}$, $\mathbf{c}$ two vectors parallel to the given lines. Then $\mathbf{b}\times\mathbf{c}$ is perpendicular to the plane; and we have only to write down the equation of the plane through $\mathbf{a}$ perpendicular to $\mathbf{b}\times\mathbf{c}$. By Art. 29 this is

$$(\mathbf{r}-\mathbf{a})\cdot\mathbf{b}\times\mathbf{c}=0,$$

that is $$\mathbf{r}\cdot\mathbf{b}\times\mathbf{c}=[\mathbf{abc}].$$

(iii) *Plane containing a given straight line and parallel to another.* Let the first line be represented by

$$\mathbf{r}=\mathbf{a}+t\mathbf{b},$$

while the second is parallel to $\mathbf{c}$. Then the plane in question

contains the point $\mathbf{a}$, and is parallel to $\mathbf{b}$ and $\mathbf{c}$. Its equation is therefore, by the last case,

$$\mathbf{r}\cdot\mathbf{b}\times\mathbf{c}=[\mathbf{abc}].$$

(iv) *Plane through two given points and parallel to a given straight line.* Let $\mathbf{a}$, $\mathbf{b}$ be the two given points, and $\mathbf{c}$ a vector parallel to the given straight line. The required plane then passes through $\mathbf{a}$ and is parallel to $\mathbf{b}-\mathbf{a}$ and $\mathbf{c}$. Its equation is therefore

$$\begin{aligned}\mathbf{r}\cdot(\mathbf{b}-\mathbf{a})\times\mathbf{c}&=[\mathbf{a},\ \mathbf{b}-\mathbf{a},\ \mathbf{c}]\\&=[\mathbf{abc}].\end{aligned}$$

(v) *Plane containing a given straight line and a given point.* Let $\mathbf{r}=\mathbf{a}+t\mathbf{b}$ be the given straight line and $\mathbf{c}$ the given point. Then the plane in question passes through the two points $\mathbf{a}$, $\mathbf{c}$ and is parallel to $\mathbf{b}$. Hence by (iv) its equation is

$$\mathbf{r}\cdot(\mathbf{a}-\mathbf{c})\times\mathbf{b}=[\mathbf{abc}].$$

49. Condition of intersection of two straight lines. Let the equations of the given straight lines be

$$\left.\begin{aligned}\mathbf{r}&=\mathbf{a}+t\mathbf{b},\\\mathbf{r}&=\mathbf{a}'+s\mathbf{b}',\end{aligned}\right\}$$

so that they pass through the points $\mathbf{a}$, $\mathbf{a}'$ and are parallel to $\mathbf{b}$, $\mathbf{b}'$ respectively. If they intersect, their common plane must be

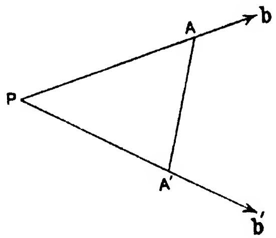

FIG. 41.

parallel to each of the vectors $\mathbf{b}$, $\mathbf{b}'$, $\mathbf{a}-\mathbf{a}'$, whose scalar triple product is therefore zero. Hence the required condition is

$$[\mathbf{b},\ \mathbf{b}',\ \mathbf{a}-\mathbf{a}']=0.$$

50. The common perpendicular to two non-intersecting straight lines. Let the equations of the two straight lines be

$$\begin{aligned}\mathbf{r}&=\mathbf{a}+t\mathbf{b},\\\mathbf{r}&=\mathbf{a}'+s\mathbf{b}'.\end{aligned}$$

Then the vector $\mathbf{n} = \mathbf{b} \times \mathbf{b}'$ is perpendicular to both lines, and therefore parallel to their common perpendicular $P'P$. If A, A' are the points $\mathbf{a}$, $\mathbf{a}'$ respectively, the length p of this common perpendicular is equal to the length of the projection of $A'A$ on $\mathbf{n}$. Hence

$$p = \frac{\mathbf{n} \cdot (\mathbf{a} - \mathbf{a}')}{n}$$
$$= \frac{1}{n}[\mathbf{b}, \mathbf{b}', \mathbf{a} - \mathbf{a}'].$$

In finding this value of p we have assumed that the direction of $\mathbf{b} \times \mathbf{b}'$ is from P' to P. This will be the case when the moment about the line $A'P'$ of the vector $\mathbf{b}$ localised in AP is positive or right-handed relative to $\mathbf{b}'$. In this case the moment about AP of the vector $\mathbf{b}'$ localised in $A'P'$ is positive relative to $\mathbf{b}$. We may adopt this convention for the sign of the perpendicular distance, so that p is positive when the moment is positive. In so doing we attach to the two lines the sense of the vectors $\mathbf{b}$ and $\mathbf{b}'$ respectively. The condition found above for the intersection of the two lines is, of course, equivalent to the vanishing of p.

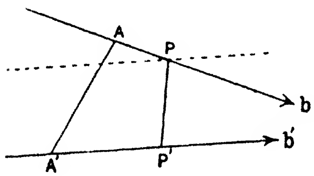

FIG. 42.

If $\mathbf{r}$ is any point on the common perpendicular to the two lines, the vectors $\mathbf{r} - \mathbf{a}$, $\mathbf{b}$ and $\mathbf{b} \times \mathbf{b}'$ are coplanar. Similarly $\mathbf{r} - \mathbf{a}'$, $\mathbf{b}'$ and $\mathbf{b} \times \mathbf{b}'$ are coplanar. Hence the straight line which cuts both the given lines at right angles is the line of intersection of the planes

$$[\mathbf{r} - \mathbf{a}, \mathbf{b}, \mathbf{b} \times \mathbf{b}'] = 0,$$
$$[\mathbf{r} - \mathbf{a}', \mathbf{b}', \mathbf{b} \times \mathbf{b}'] = 0.$$

The *equations of two non-intersecting straight lines* can be put in a convenient form by choosing as origin the middle point of their common perpendicular. The equations of the lines are then

$$\mathbf{r} = \mathbf{c} + t\mathbf{b}, \quad \mathbf{r} = -\mathbf{c} + s\mathbf{b}',$$

where

$$\mathbf{c} = \tfrac{1}{2}\overrightarrow{P'P} = \tfrac{1}{2}p\hat{\mathbf{n}} = \tfrac{1}{2}p\frac{\mathbf{n}}{n}$$
$$= \frac{1}{2n^2}[\mathbf{b}, \mathbf{b}', \mathbf{a} - \mathbf{a}'](\mathbf{b} \times \mathbf{b}').$$

The vector $\mathbf{c}$, being perpendicular to both lines, satisfies the equations

$$\mathbf{c}\cdot\mathbf{b} = \mathbf{c}\cdot\mathbf{b}' = 0.$$

Example.

When a ray of light is reflected from a plane mirror, the shortest distance between the incident ray and any straight line on the mirror is equal to that between the reflected ray and the same straight line.

Let the unit vector in the direction of the incident ray be expressed in the form $\mathbf{a} - \mathbf{b}$, where $\mathbf{a}$ is parallel to the mirror and $\mathbf{b}$ perpendicular to it. Then the unit vector in the direction of the reflected ray is $\mathbf{a} + \mathbf{b}$, because reflection at the mirror reverses the direction of the normal component, but leaves the other unchanged. Hence if the point of incidence is taken as origin the incident and reflected rays are

$$\mathbf{r} = t(\mathbf{a} - \mathbf{b}) \qquad \text{.........(1)}$$

and

$$\mathbf{r} = t(\mathbf{a} + \mathbf{b}), \qquad \text{.........(2)}$$

respectively. Any straight line in the plane of the mirror may be represented by

$$\mathbf{r} = \mathbf{c} + t\mathbf{d}, \qquad \text{.........(3)}$$

where $\mathbf{c}$ and $\mathbf{d}$ are perpendicular to $\mathbf{b}$. The shortest distance between (1) and (3) is

$$\frac{[\mathbf{a} - \mathbf{b},\ \mathbf{d},\ -\mathbf{c}]}{|(\mathbf{a} - \mathbf{b})\times\mathbf{d}|}.$$

But $\mathbf{a}$, $\mathbf{d}$, $\mathbf{c}$ are coplanar, so that the numerator is equal to $[\mathbf{bdc}]$, which merely changes sign when $\mathbf{b}$ is replaced by $-\mathbf{b}$. And since $\mathbf{a}\times\mathbf{d}$ is perpendicular to $\mathbf{b}\times\mathbf{d}$, the denominator has the value

$$\sqrt{(\mathbf{a}\times\mathbf{d})^2 + (\mathbf{b}\times\mathbf{d})^2},$$

which remains unchanged when $\mathbf{b}$ is replaced by $-\mathbf{b}$. Hence the result.

51. Plücker's coordinates of a straight line. Let $\mathbf{d}$ be a *unit* vector parallel to a given line, and $\mathbf{m}$ the moment about the origin of the vector $\mathbf{d}$ localised in the given line; so that

$$\mathbf{m} = \mathbf{r}\times\mathbf{d},$$

where $\mathbf{r}$ is the position vector of any point on the line. Then the position of the line is uniquely specified by $\mathbf{d}$ and $\mathbf{m}$. The direction of the line is that of $\mathbf{d}$. And since $\mathbf{m}$ is perpendicular to the plane containing the origin and the given line, the direction of $\mathbf{m}$ determines a plane in which the straight line must lie, and the magnitude of $\mathbf{m}$ determines its distance from the origin. Thus, if $\mathbf{d}$, $\mathbf{m}$ are given the position of the straight line is given. The six quantities known as Plücker's coordinates of the line

are the (scalar) resolutes of **d**, **m** along rectangular axes through O.

Let **d**, **m** and **d**$'$, **m**$'$ be the Plücker's coordinates of two lines, and **r**, **r**$'$ the position vectors of two points P, P', one on each line.

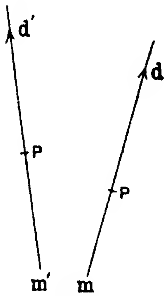

FIG. 43.

Then the moment about P' of the unit vector **d** localised in the first line is $(\mathbf{r}-\mathbf{r}')\times\mathbf{d}$, and the moment of **d** about the second line is therefore, by Art. 40,

$$(\mathbf{r}-\mathbf{r}')\times\mathbf{d}\cdot\mathbf{d}' = (\mathbf{r}\times\mathbf{d})\cdot\mathbf{d}' + (\mathbf{r}'\times\mathbf{d}')\cdot\mathbf{d}$$
$$= \mathbf{m}\cdot\mathbf{d}' + \mathbf{m}'\cdot\mathbf{d}.$$

This result is symmetrical, and represents the moment about either line of a unit vector localised in the other. It is called the *mutual moment* of the two lines, with senses the same as **d**, **d**$'$ respectively.

Since **d**, **d**$'$ are unit vectors, the module of $\mathbf{d}\times\mathbf{d}'$ is $\sin\theta$, where θ is the angle of inclination of the two lines. Then, by Art. 50, the length of the common perpendicular to the two lines is

$$p = \frac{[\mathbf{d}, \mathbf{d}', \mathbf{r}-\mathbf{r}']}{\sin\theta} = \frac{M}{\sin\theta},$$

where M is their mutual moment. Hence the mutual moment of the lines is given by

$$M = p\sin\theta.$$

Examples.

(1) *Find Plucker's coordinates for the line of intersection of the planes* $\mathbf{r}\cdot\mathbf{n} = q$ *and* $\mathbf{r}\cdot\mathbf{n}' = q'$.

The line of intersection is perpendicular to both **n** and **n**$'$, so that

$$\mathbf{d} = \frac{\mathbf{n}\times\mathbf{n}'}{N},$$

where $N = \text{mod } \mathbf{n}\times\mathbf{n}'$. If **b** is any point on the line, **b** lies on both planes, so that

$$\mathbf{b}\cdot\mathbf{n} = q \quad \text{and} \quad \mathbf{b}\cdot\mathbf{n}' = q'.$$

Then
$$\mathbf{m} = \mathbf{b}\times\mathbf{d} = \mathbf{b}\times(\mathbf{n}\times\mathbf{n}')/N$$
$$= \frac{1}{N}(\mathbf{b}\cdot\mathbf{n}'\mathbf{n} - \mathbf{b}\cdot\mathbf{n}\mathbf{n}')$$
$$= \frac{q'\mathbf{n} - q\mathbf{n}'}{N}.$$

(2) *Find the equation of the plane through the line* **d**, **m** *parallel to* **c**.

Since the required plane is parallel to **d** and **c**, its normal is parallel to $\mathbf{d}\times\mathbf{c}$. If **b** is a point on the given line

$$\mathbf{b}\times\mathbf{d}=\mathbf{m},$$

and, **r** being a current point on the plane, $\mathbf{r}-\mathbf{b}$ is parallel to the plane, and therefore perpendicular to the normal. Hence

$$(\mathbf{r}-\mathbf{b})\cdot(\mathbf{d}\times\mathbf{c})=0,$$

which may be written $\quad [\mathbf{rdc}]=\mathbf{m}\cdot\mathbf{c}.$

This is the required equation of the plane.

(3) *Find the equation of the plane through the point* **a** *and the line* **d**, **m**.

If **b** is a point on the given line $\mathbf{b}\times\mathbf{d}=\mathbf{m}$. The plane is parallel to $\mathbf{b}-\mathbf{a}$ and **d**, and its normal is parallel to $(\mathbf{b}-\mathbf{a})\times\mathbf{d}$. Hence if **r** is a current point on the plane $(\mathbf{r}-\mathbf{a})$ is also parallel to the plane, and therefore

$$(\mathbf{r}-\mathbf{a})\cdot(\mathbf{b}-\mathbf{a})\times\mathbf{d}=0,$$

which may be written $\quad \mathbf{r}\cdot\mathbf{m}-[\mathbf{rad}]=\mathbf{a}\cdot\mathbf{m}.$

√**52. Volume of a tetrahedron.** With one vertex O as origin, let the other vertices A, B, C be the points **a**, **b**, **c** respectively. Then the vector area of OBC is $\frac{1}{2}\mathbf{b}\times\mathbf{c}$, and the volume of the tetrahedron is

$$V=\tfrac{1}{3}\mathbf{a}\cdot(\tfrac{1}{2}\mathbf{b}\times\mathbf{c})$$
$$=\tfrac{1}{6}[\mathbf{abc}].$$

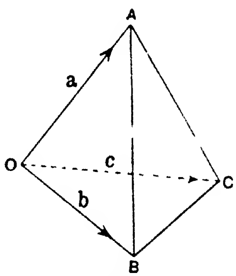

FIG. 44.

Suppose we require the length p of the common perpendicular to the two edges AB, OC. The directions of these lines are those of the vectors $\mathbf{b}-\mathbf{a}$ and **c**, while **a**, **c** are two points, one on each line. Hence, by Art. 50, if θ is their angle of inclination,

$$p=\frac{[\mathbf{b}-\mathbf{a},\ \mathbf{c},\ \mathbf{a}-\mathbf{c}]}{AB\,.\,OC\,.\sin\theta}.$$

The numerator of this expression reduces to $[\mathbf{abc}]$ or $6V$. Hence the relation

$$V=\tfrac{1}{6}AB\,.\,OC\,.\,p\,.\sin\theta$$
$$=\tfrac{1}{6}AB\,.\,OC\,.\,M,$$

where M is the mutual moment of the straight lines along the edges.

Cor. The volume of a tetrahedron whose vertices are the points $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ is $\frac{1}{6}[\mathbf{a}-\mathbf{d},\ \mathbf{b}-\mathbf{d}.\ \mathbf{c}-\mathbf{d}]$, which reduces to

$$\tfrac{1}{6}\{[\mathbf{abc}]-[\mathbf{abd}]+[\mathbf{acd}]-[\mathbf{bcd}]\}.$$

Spherical Trigonometry.

53. Two fundamental formulae. Let A, B, C be points on the surface of a sphere of unit radius, whose position vectors relative to the centre O are the unit vectors $\mathbf{l}$, $\mathbf{m}$, $\mathbf{n}$ respectively. If these points are joined by arcs of great circles, the figure so formed is a spherical triangle. The sides a, b, c of this triangle are the angles BOC, COA, AOB which these arcs subtend at the centre. The angle A of the triangle is the angle between the planes AOB and AOC; and the other angles are similarly interpreted.

Consider the expansion

$$(\mathbf{l}\times\mathbf{m})\cdot(\mathbf{l}\times\mathbf{n}) = \mathbf{l}\cdot\mathbf{l}\,\mathbf{m}\cdot\mathbf{n} - \mathbf{l}\cdot\mathbf{n}\,\mathbf{l}\cdot\mathbf{m}$$

as applied to the spherical triangle. Since the vectors are all unit vectors, $\mathbf{m}\cdot\mathbf{n}=\cos a$, $\mathbf{n}\cdot\mathbf{l}=\cos b$, and $\mathbf{l}\cdot\mathbf{m}=\cos c$. Further, $\mathbf{l}\times\mathbf{m}$ is a vector whose module is $\sin c$, and whose direction is perpendicular to the plane AOB drawn inward. Similarly $\mathbf{l}\times\mathbf{n}$ has a module $\sin b$, and a direction perpendicular to the plane AOC drawn outward. Hence

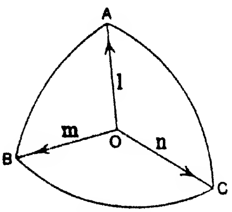

FIG. 45.

$$(\mathbf{l}\times\mathbf{m})\cdot(\mathbf{l}\times\mathbf{n}) = \sin b \sin c \cos A.$$

The above expansion then gives the relation

$$\sin b \sin c \cos A = \cos a - \cos b \cos c.$$

This is one of the fundamental formulae of spherical trigonometry; and two similar ones may be written down by cyclic permutation of the sides and angles. They are usually put in the form

$$\cos a = \cos b \cos c + \sin b \sin c \cos A,$$
$$\cos b = \cos c \cos a + \sin c \sin a \cos B,$$
$$\cos c = \cos a \cos b + \sin a \sin b \cos C.$$

Another formula for the spherical triangle may be deduced from the equality

$$(\mathbf{l}\times\mathbf{m})\times(\mathbf{l}\times\mathbf{n}) = [\mathbf{lmn}]\mathbf{l}.$$

For, $\mathbf{l}\times\mathbf{m}$ and $\mathbf{l}\times\mathbf{n}$ having the meanings already stated, their cross product is a vector perpendicular to the normals to the planes AOB and AOC, and therefore in the direction of $\mathbf{l}$; while the module of the vector is $\sin c \sin b \sin A$. The above equation shows that

$$\sin c \sin b \sin A \,.\, \mathbf{l} = [\mathbf{lmn}]\mathbf{l},$$

that is

$$\sin b \sin c \sin A = [\mathbf{lmn}].$$

Cyclic permutation of the sides and angles shows that

$$\sin c \sin a \sin B \quad \text{and} \quad \sin a \sin b \sin C$$

have the same value. From this it follows that

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c},$$

which is another fundamental formula.

Concurrent Forces.

54. Four forces in equilibrium. If four forces acting at a point are in equilibrium, their vector sum is zero. Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ be *unit* vectors in the directions of the forces, and $F_1, \ldots, F_4$ the measures of the forces. Then

$$F_1\mathbf{a} + F_2\mathbf{b} + F_3\mathbf{c} + F_4\mathbf{d} = 0. \quad \text{.............................(1)}$$

If we multiply throughout scalarly by $\mathbf{c}\times\mathbf{d}$, two of the terms disappear containing triple products with a repeated factor, and we find

$$F_1[\mathbf{acd}] + F_2[\mathbf{bcd}] = 0.$$

Similarly, on forming the scalar product of the first member of (1) with $\mathbf{a}\times\mathbf{c}$, we have

$$F_2[\mathbf{bac}] + F_4[\mathbf{acd}] = 0.$$

From these and a third equation which may be derived in the same way, we find

$$\frac{F_1}{[\mathbf{bcd}]} = \frac{F_2}{[\mathbf{cad}]} = \frac{F_3}{[\mathbf{abd}]} = \frac{-F_4}{[\mathbf{abc}]}.$$

Thus each force is proportional to the scalar triple product of unit vectors in the directions of the other three, and therefore to the volume of the parallelepiped determined by those vectors. This theorem is usually attributed to Rankine.

Lami's theorem for the equilibrium of three forces, already considered in Art. 13, may be proved by a similar method. For if three forces are in equilibrium,

$$F_1\mathbf{a} + F_2\mathbf{b} + F_3\mathbf{c} = 0.$$

On cross multiplication by $\mathbf{a}$, we find

$$F_2\mathbf{b}\times\mathbf{a}+F_3\mathbf{c}\times\mathbf{a}=0,$$

and similarly

$$F_1\mathbf{b}\times\mathbf{a}+F_3\mathbf{b}\times\mathbf{c}=0.$$

From the last two equations it follows that

$$\frac{\mathbf{b}\times\mathbf{c}}{F_1}=\frac{\mathbf{c}\times\mathbf{a}}{F_2}=\frac{\mathbf{a}\times\mathbf{b}}{F_3}.$$

Since their cross products are parallel, the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are coplanar, and each force is proportional to the sine of the angle between the other two.

EXERCISES ON CHAPTER IV.

1. Show that $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})+\mathbf{b}\times(\mathbf{c}\times\mathbf{a})+\mathbf{c}\times(\mathbf{a}\times\mathbf{b})=0.$

2. Prove the relation

$$\mathbf{a}\times\{\mathbf{b}\times(\mathbf{c}\times\mathbf{d})\}=\mathbf{b}\cdot\mathbf{d}\;\mathbf{a}\times\mathbf{c}-\mathbf{b}\cdot\mathbf{c}\;\mathbf{a}\times\mathbf{d},$$

and hence expand $\mathbf{a}\times(\mathbf{b}\times\{\mathbf{c}\times(\mathbf{d}\times\mathbf{e})\}).$

3. Show that $[\mathbf{a}\times\mathbf{b}\;\mathbf{b}\times\mathbf{c}\;\mathbf{c}\times\mathbf{a}]=[\mathbf{abc}]^2,$

and express the result by means of determinants.

4. Prove that

$$[\mathbf{lmn}][\mathbf{abc}]=\begin{vmatrix}\mathbf{l}\cdot\mathbf{a} & \mathbf{l}\cdot\mathbf{b} & \mathbf{l}\cdot\mathbf{c}\\ \mathbf{m}\cdot\mathbf{a} & \mathbf{m}\cdot\mathbf{b} & \mathbf{m}\cdot\mathbf{c}\\ \mathbf{n}\cdot\mathbf{a} & \mathbf{n}\cdot\mathbf{b} & \mathbf{n}\cdot\mathbf{c}\end{vmatrix},$$

and give its Cartesian equivalent.

5. Find the equation of the plane which contains the line $\mathbf{r}=t\mathbf{a}$, and is perpendicular to the plane containing $\mathbf{r}=u\mathbf{b}$ and $\mathbf{r}=v\mathbf{c}$.

6. Find the equation of the plane containing the two parallel lines

$$\mathbf{r}=\mathbf{a}+s\mathbf{b},\quad \mathbf{r}=\mathbf{a}'+t\mathbf{b}.$$

7. What is the equation of the plane containing the line $\mathbf{r}-\mathbf{a}=t\mathbf{b}$, and perpendicular to the plane $\mathbf{r}\cdot\mathbf{c}=q$?

8. Show that the plane containing the two straight lines $\mathbf{r}-\mathbf{a}=t\mathbf{a}'$ and $\mathbf{r}-\mathbf{a}'=s\mathbf{a}$ is represented by

$$[\mathbf{raa}']=0.$$

Give a geometrical interpretation.

9. The shortest distances between a diagonal of a rectangular parallelepiped, whose sides are a, b, c, and the edges not meeting it, are

$$\frac{bc}{\sqrt{b^2+c^2}},\quad \frac{ca}{\sqrt{c^2+a^2}},\quad \frac{ab}{\sqrt{a^2+b^2}}.$$

10. The shortest distance between two opposite edges of a regular tetrahedron is equal to half the diagonal of the square described on an edge.

11. Find the shortest distance between the straight lines $\mathbf{r}=t\mathbf{k}$ and $\mathbf{r}-\mathbf{a}=s\mathbf{b}$, and determine the equation of the line which cuts both at right angles.

12. Show that the perpendicular distance of the point $\mathbf{a}$ from the line whose Plücker's coordinates are $\mathbf{d}$, $\mathbf{m}$ is mod $(\mathbf{m}+\mathbf{d}\times\mathbf{a})$.

13. Find the equation of the straight line through the point $\mathbf{c}$, intersecting both the lines $\mathbf{r}-\mathbf{a}=s\mathbf{b}$ and $\mathbf{r}-\mathbf{a}'=t\mathbf{b}'$.

14. Find the straight line through the point $\mathbf{c}$, which is parallel to the plane $\mathbf{r}\cdot\mathbf{a}=0$, and intersects the line $\mathbf{r}-\mathbf{a}'=t\mathbf{b}$.

15. A straight line intersects two non-coplanar straight lines, and moves parallel to a fixed plane. Prove that the locus of a point which divides the intercept in a constant ratio is a straight line.

16. The locus of the middle points of all straight lines terminated by two fixed non-intersecting straight lines is a plane bisecting their common perpendicular at right angles.

17. Find the locus of a point which is equidistant from the three planes $\mathbf{r}\cdot\mathbf{n}_1=q_1$, $\mathbf{r}\cdot\mathbf{n}_2=q_2$, $\mathbf{r}\cdot\mathbf{n}_3=q_3$.

18. Show that the six planes, each passing through one edge of a tetrahedron and bisecting the opposite edge, meet in a point.

19. The six planes bisecting the angles between consecutive faces of a tetrahedron meet in a point.

20. Prove the formulae

$$\begin{aligned}[\mathbf{a}\times\mathbf{b},\ \mathbf{c}\times\mathbf{d},\ \mathbf{e}\times\mathbf{f}] &= [\mathbf{abd}][\mathbf{cef}]-[\mathbf{abc}][\mathbf{def}]\\ &= [\mathbf{abe}][\mathbf{fcd}]-[\mathbf{abf}][\mathbf{ecd}]\\ &= [\mathbf{cda}][\mathbf{bef}]-[\mathbf{cdb}][\mathbf{aef}].\end{aligned}$$

21. Show that the volume of the tetrahedron bounded by the four planes

$\mathbf{r}\cdot(m\mathbf{j}+n\mathbf{k})=0,\ \ \mathbf{r}\cdot(n\mathbf{k}+l\mathbf{i})=0,\ \ \mathbf{r}\cdot(l\mathbf{i}+m\mathbf{j})=0$ and $\mathbf{r}\cdot(l\mathbf{i}+m\mathbf{j}+n\mathbf{k})=p$

is $2p^3/3lmn$.

22. Prove that the four points $4\mathbf{i}+5\mathbf{j}+\mathbf{k}$, $-(\mathbf{j}+\mathbf{k})$, $3\mathbf{i}+9\mathbf{j}+4\mathbf{k}$ and $4(-\mathbf{i}+\mathbf{j}+\mathbf{k})$ are coplanar.

23. If a straight line is drawn in each face of any trihedral angle, through the vertex and perpendicular to the third edge, the three lines thus drawn are coplanar.

✓ **24.** Prove the formula

$$(\mathbf{b}\times\mathbf{c})\cdot(\mathbf{a}\times\mathbf{d})+(\mathbf{c}\times\mathbf{a})\cdot(\mathbf{b}\times\mathbf{d})+(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})=0,$$

and use it to show that

$$\begin{aligned}\sin(A+B)\sin(A-B)&=\sin^2 A-\sin^2 B\\&=\tfrac{1}{2}(\cos 2B-\cos 2A).\end{aligned}$$

25. The scalar moment about the line CD, of a force represented by the localised vector $\overrightarrow{AB}$, is $6V/l$, where l is the length of CD and V the volume of the tetrahedron $ABCD$.

26. If in a tetrahedron the mutual moment of the opposite edges is the same for each pair, prove that the product of their lengths is also the same for each pair.

27. From the Cor. to Art. 52 show that the volume of a tetrahedron is given, in terms of the coordinates of its vertices, by the determinant

$$\frac{1}{6}\begin{vmatrix} a_1-d_1 & a_2-d_2 & a_3-d_3\\ b_1-d_1 & b_2-d_2 & b_3-d_3\\ c_1-d_1 & c_2-d_2 & c_3-d_3\end{vmatrix}=\frac{1}{6}\begin{vmatrix} a_1 & a_2 & a_3 & 1\\ b_1 & b_2 & b_3 & 1\\ c_1 & c_2 & c_3 & 1\\ d_1 & d_2 & d_3 & 1\end{vmatrix}$$

✓ **28.** Prove the following formula for the volume V of a tetrahedron, in terms of the lengths of three concurrent edges and their mutual inclinations:

$$V^2=\frac{a^2b^2c^2}{36}\begin{vmatrix} 1 & \cos\phi & \cos\psi\\ \cos\phi & 1 & \cos\theta\\ \cos\psi & \cos\theta & 1\end{vmatrix}$$

29. If the four lines joining corresponding vertices of two tetrahedra are concurrent, the lines of intersection of corresponding faces are coplanar; and conversely.

30. The condition of intersection of the two lines whose Plücker's coordinates are $\mathbf{d}$, $\mathbf{m}$ and $\mathbf{d}'$, $\mathbf{m}'$ is $\mathbf{d}\cdot\mathbf{m}'+\mathbf{d}'\cdot\mathbf{m}=0$.

31. Find Plücker's coordinates for the straight line which cuts each of the lines $\mathbf{d}$, $\mathbf{m}$ and $\mathbf{d}'$, $\mathbf{m}'$ at right angles.

32. If the mutual direction cosines of two sets of rectangular unit vectors are given by the accompanying table, show that the sum of the squares of the terms in any row or column is equal to unity; that the sum of the products of corresponding terms in any two rows or any two columns is zero; and that the determinant of nine terms has the value unity.

	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{i}'$	l_1	m_1	n_1
$\mathbf{j}'$	l_2	m_2	n_2
$\mathbf{k}'$	l_3	m_3	n_3

CHAPTER V

DIFFERENTIATION AND INTEGRATION OF VECTORS CURVATURE AND TORSION OF CURVES.

55. Our object in the present chapter is to explain the differentiation and integration of vectors with respect to a scalar variable. Only cases of one independent variable will be considered, partial differentiation being left for the second volume. Within the limits of a single chapter it is impossible to examine fully the points that arise in connection with continuity and the existence of a limit. We therefore lay no claim to rigour of the kind that is necessary in a treatise on the infinitesimal calculus. Our object is to *explain* rather than to prove.

Derivative of a vector. Suppose that a vector $\mathbf{r}$ is a continuous and single-valued function of a scalar variable t. Then to each value of t corresponds only one value of $\mathbf{r}$; and as t varies continuously, so does $\mathbf{r}$. Relative to a fixed origin O, let P be the point whose position vector is $\mathbf{r}$. Then if t varies continuously, P moves along a continuous curve in space.

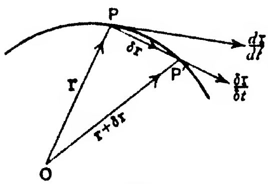

FIG. 46.

Let $\overrightarrow{OP}$ be the value of $\mathbf{r}$ corresponding to the value t of the scalar variable. An increment δt in the latter will produce an increment $\delta\mathbf{r}$ in the former. Thus the value $t+\delta t$ of the scalar corresponds to the value $\mathbf{r}+\delta\mathbf{r}$ of the vector, and this is the position vector of another point P' on the curve. The increment $\delta\mathbf{r}$ is equal to the vector $\overrightarrow{PP'}$. The quotient $\frac{\delta\mathbf{r}}{\delta t}$, of the vector $\delta\mathbf{r}$

by the number δt, is itself a vector. If now δt is a small increment, $\delta\mathbf{r}$ will also in general be small. As δt tends to the value zero, the point P' moves up to coincidence with P, and the chord PP' to coincidence with the tangent at P to the curve. The limiting value of the quotient $\frac{\delta\mathbf{r}}{\delta t}$, as δt tends to zero, is a vector whose direction is the limiting direction of $\delta\mathbf{r}$, which is the direction of the tangent at P. This limiting value of the quotient, when it exists, is called the *derivative* or *differential coefficient* of $\mathbf{r}$ with respect to t, and is denoted by $\frac{d\mathbf{r}}{dt}$. Thus

$$\frac{d\mathbf{r}}{dt} = \operatorname*{Lt}_{\delta t \to 0} \frac{\delta\mathbf{r}}{\delta t}.$$

The process of determining the derivative is called *differentiation*.

The derivative of $\mathbf{r}$ is also in general a function of t, and itself possesses a derivative which is called the second derivative of $\mathbf{r}$, and is denoted by $\frac{d^2\mathbf{r}}{dt^2}$. Similarly the derivative of this is called the third derivative of $\mathbf{r}$, and is denoted by $\frac{d^3\mathbf{r}}{dt^3}$.

A case of special importance is that in which t is the time variable, and $\mathbf{r}$ the position vector of a moving particle P relative to the origin O. Then $\delta\mathbf{r}$ represents the displacement of the particle during the interval δt, and therefore $\frac{\delta\mathbf{r}}{\delta t}$ the average velocity during that interval. The limiting value of this average velocity as δt tends to zero is the *instantaneous velocity* of the particle. Hence the vector $\mathbf{v}$ representing the instantaneous velocity of P is

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}.$$

This vector is, of course, in the direction of the tangent to the path of the particle. Similarly, if $\delta\mathbf{v}$ is the increment in the velocity vector $\mathbf{v}$ during the interval δt, the quotient $\frac{\delta\mathbf{v}}{\delta t}$ represents the average acceleration during that interval. The *instantaneous acceleration* of the particle is the limiting value of this average acceleration as δt tends to zero. Thus the vector

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2}$$

represents the instantaneous acceleration of the moving particle.

The derivative of any *constant vector* $\mathbf{c}$ is zero; for the increment δt produces no change in $\mathbf{c}$.

The *derivative of the sum* $\mathbf{r}+\mathbf{s}$ of two vectors $\mathbf{r}$ and $\mathbf{s}$, which are both functions of t, is equal to the sum of their derivatives. For if $\delta\mathbf{r}$ and $\delta\mathbf{s}$ are the increments in these vectors due to the increment δt,

$$\begin{aligned}\delta(\mathbf{r}+\mathbf{s}) &= (\mathbf{r}+\delta\mathbf{r}+\mathbf{s}+\delta\mathbf{s})-(\mathbf{r}+\mathbf{s})\\ &= \delta\mathbf{r}+\delta\mathbf{s},\end{aligned}$$

and therefore the quotient

$$\frac{\delta(\mathbf{r}+\mathbf{s})}{\delta t} = \frac{\delta\mathbf{r}}{\delta t}+\frac{\delta\mathbf{s}}{\delta t}.$$

Taking limiting values of both sides as δt tends to zero, we have

$$\frac{d}{dt}(\mathbf{r}+\mathbf{s}) = \frac{d\mathbf{r}}{dt}+\frac{d\mathbf{s}}{dt}.$$

The argument is obviously true for the sum of any number of vectors.

Suppose that $\mathbf{r}$ is a continuous function of a scalar variable s, and s a continuous function of another, t. Then an increment δt in the last produces increments $\delta\mathbf{r}$, δs in the others, which both tend to zero with δt. The relation

$$\frac{\delta\mathbf{r}}{\delta t} = \frac{\delta\mathbf{r}}{\delta s}\frac{\delta s}{\delta t}$$

is an algebraical identity, the number placed after the vector having the same meaning as when placed in front. Taking limiting values of both sides as δt tends to zero we have the formula

$$\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds}\frac{ds}{dt},$$

as in algebraic calculus.

56. Derivatives of products. The derivative of any product of vectors is found in the same way as for an algebraic product, being equal to the sum of the quantities got by differentiating a single factor and leaving the others unchanged.

Take for instance the product $u\mathbf{r}$ of a scalar u and a vector $\mathbf{r}$, both functions of the variable t. If δu and $\delta\mathbf{r}$ are their increments due to the increment δt, the increment in the product is

$$\begin{aligned}\delta(u\mathbf{r}) &= (u+\delta u)(\mathbf{r}+\delta\mathbf{r})-u\mathbf{r}\\ &= \delta u\,\mathbf{r}+u\,\delta\mathbf{r}+\delta u\,\delta\mathbf{r}.\end{aligned}$$

Dividing throughout by δt, we have

$$\frac{\delta(u\mathbf{r})}{\delta t} = \frac{\delta u}{\delta t}\mathbf{r} + u\frac{\delta \mathbf{r}}{\delta t} + \frac{\delta u}{\delta t}\delta\mathbf{r},$$

and on taking limiting values on both sides,

$$\frac{d}{dt}(u\mathbf{r}) = \frac{du}{dt}\mathbf{r} + u\frac{d\mathbf{r}}{dt}. \quad \text{.............................(1)}$$

From this and the preceding results it follows that, if any vector $\mathbf{r}$ is expressed as the sum of rectangular components by the formula

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k},$$

then, since $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are constant vectors, the derivative of $\mathbf{r}$ is

$$\frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k},$$

and similarly for the second and higher derivatives.

The same argument as above shows that the derivatives of the scalar and vector products $\mathbf{r}\cdot\mathbf{s}$ and $\mathbf{r}\times\mathbf{s}$ are given by the formulae

$$\frac{d}{dt}(\mathbf{r}\cdot\mathbf{s}) = \frac{d\mathbf{r}}{dt}\cdot\mathbf{s} + \mathbf{r}\cdot\frac{d\mathbf{s}}{dt} \quad \text{............................(2)}$$

and

$$\frac{d}{dt}(\mathbf{r}\times\mathbf{s}) = \frac{d\mathbf{r}}{dt}\times\mathbf{s} + \mathbf{r}\times\frac{d\mathbf{s}}{dt}, \quad \text{..............................(3)}$$

it being understood that, in the last formula, the order of the factors in any term must not be changed unless the sign is changed at the same time. To prove (2) we have

$$\begin{aligned}\delta(\mathbf{r}\cdot\mathbf{s}) &= (\mathbf{r}+\delta\mathbf{r})\cdot(\mathbf{s}+\delta\mathbf{s}) - \mathbf{r}\cdot\mathbf{s}\\ &= \delta\mathbf{r}\cdot\mathbf{s} + \mathbf{r}\cdot\delta\mathbf{s} + \delta\mathbf{r}\cdot\delta\mathbf{s}.\end{aligned}$$

Dividing throughout by δt and proceeding to the limit, we obtain the required result.

If in (2) we take $\mathbf{s}$ equal to $\mathbf{r}$, we find the useful formula

$$\frac{d}{dt}(\mathbf{r}\cdot\mathbf{r}) = \frac{d\mathbf{r}^2}{dt} = 2\mathbf{r}\cdot\frac{d\mathbf{r}}{dt}.$$

But if r is the module of $\mathbf{r}$, the product $\mathbf{r}\cdot\mathbf{r} = \mathbf{r}^2 = r^2$, and the derivative of r^2 is $2r\dfrac{dr}{dt}$. Thus

$$\mathbf{r}\cdot\frac{d\mathbf{r}}{dt} = r\frac{dr}{dt}.$$

In the case of a vector $\mathbf{a}$ of constant length,

$$\mathbf{a}^2 = a^2 = \text{a constant},$$

and therefore
$$\mathbf{a}\cdot\frac{d\mathbf{a}}{dt}=0,$$
showing that the derivative $\frac{d\mathbf{a}}{dt}$ is perpendicular to $\mathbf{a}$. A point whose position vector is $\mathbf{a}$ then lies on the surface of a sphere of radius a, and the vector $\frac{d\mathbf{a}}{dt}$ is parallel to the tangent plane at the point.

If in (3) we replace $\mathbf{s}$ by $\frac{d\mathbf{r}}{dt}$, we find the formula
$$\frac{d}{dt}\left(\mathbf{r}\times\frac{d\mathbf{r}}{dt}\right)=\mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2},$$
since the cross product of two equal vectors $\frac{d\mathbf{r}}{dt}$ is zero. This result is frequently useful.

Triple products are differentiated on the same principle as above. Thus, if $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ are functions of t, their scalar and vector triple products have derivatives given by
$$\frac{d}{dt}[\mathbf{pqr}]=\left[\frac{d\mathbf{p}}{dt}\mathbf{qr}\right]+\left[\mathbf{p}\frac{d\mathbf{q}}{dt}\mathbf{r}\right]+\left[\mathbf{pq}\frac{d\mathbf{r}}{dt}\right]$$
and
$$\frac{d}{dt}\mathbf{p}\times(\mathbf{q}\times\mathbf{r})=\frac{d\mathbf{p}}{dt}\times(\mathbf{q}\times\mathbf{r})+\mathbf{p}\times\left(\frac{d\mathbf{q}}{dt}\times\mathbf{r}\right)+\mathbf{p}\times\left(\mathbf{q}\times\frac{d\mathbf{r}}{dt}\right),$$
the order of the factors being maintained in each term of the second formula, and the cyclic order in each term of the first. The formulae are easily proved by a double application of (2) and (3), or from first principles as above.

57. Integration. Having given one vector $\mathbf{r}$, the process of finding another vector $\mathbf{F}$ whose derivative with respect to t is equal to $\mathbf{r}$, is called *integration*. Thus integration is the reverse process to differentiation. The vector $\mathbf{F}$ is called the *integral* of $\mathbf{r}$ with respect to t, and is written
$$\mathbf{F}=\int\mathbf{r}\,dt.$$
The symbol $\int$ is called the *integral sign*, t is the variable of integration, and the function $\mathbf{r}$ to be integrated is the *integrand*. The relation just written is equivalent to
$$\frac{d\mathbf{F}}{dt}=\mathbf{r},$$
so that the derivative of the integral is equal to the integrand.

The integral $\mathbf{F}$ is indefinite to the extent of an additive arbitrary constant vector $\mathbf{c}$. For if the derivative of $\mathbf{F}$ is equal to $\mathbf{f}$, so is the derivative of $\mathbf{F}+\mathbf{c}$. For this reason $\mathbf{F}$ is called the *indefinite integral*. The arbitrary constant $\mathbf{c}$ is termed the *constant of integration*. In the application of integration to a definite problem, the value of $\mathbf{c}$ will, as a rule, be determined from some initial or geometrical condition to be satisfied.

From the results of the preceding Art. we may write down the values of the following integrals, which illustrate the process of integration, and will subsequently be found useful :

$$\int\left(\mathbf{r}\cdot\frac{d\mathbf{s}}{dt}+\mathbf{s}\cdot\frac{d\mathbf{r}}{dt}\right)dt=\mathbf{r}\cdot\mathbf{s}+c,$$

$$\int 2\mathbf{r}\cdot\frac{d\mathbf{r}}{dt}\,dt=\mathbf{r}\cdot\mathbf{r}+c=\mathbf{r}^2+c,$$

$$\int 2\frac{d\mathbf{r}}{dt}\cdot\frac{d^2\mathbf{r}}{dt^2}\,dt=\frac{d\mathbf{r}}{dt}\cdot\frac{d\mathbf{r}}{dt}+c=\left(\frac{d\mathbf{r}}{dt}\right)^2+c,$$

$$\int\mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2}\,dt=\mathbf{r}\times\frac{d\mathbf{r}}{dt}+\mathbf{c},$$

$$\int\left(\frac{1}{r}\frac{d\mathbf{r}}{dt}-\frac{dr}{dt}\frac{\mathbf{r}}{r^2}\right)dt=\frac{\mathbf{r}}{r}+\mathbf{c}=\hat{\mathbf{r}}+\mathbf{c},$$

and if $\mathbf{a}$ is a constant vector

$$\int\mathbf{a}\times\frac{d\mathbf{r}}{dt}\,dt=\mathbf{a}\times\mathbf{r}+\mathbf{c}.$$

The constant of integration is of the same nature as the integrand. In the first three of the above results it is therefore a scalar; in the last three a vector.

As a further example in integration, suppose we are given the relation

$$\frac{d^2\mathbf{r}}{dt^2}=-n^2\mathbf{r},$$

from which it is required to find the value of $\frac{d\mathbf{r}}{dt}$. Now we cannot write down the integral of the second member, since $\mathbf{r}$ is not a known function of t. But on forming the scalar product of each side with $2\frac{d\mathbf{r}}{dt}$, we have the equation

$$2\frac{d\mathbf{r}}{dt}\cdot\frac{d^2\mathbf{r}}{dt^2}=-2n^2\mathbf{r}\cdot\frac{d\mathbf{r}}{dt},$$

which can be integrated as it stands, giving

$$\left(\frac{d\mathbf{r}}{dt}\right)^2 = c - n^2\mathbf{r}^2,$$

from which the value of $\frac{d\mathbf{r}}{dt}$ is found in terms of r.

Curvature and Torsion of a Curve.

58. Tangent at a given point. Let s be (the measure of) the length of the arc from a fixed point A on a given curve up to the variable point P. Then the position vector $\mathbf{r}$ of P, relative to a fixed origin O, is a function of the scalar variable s. We shall consider only curves for which $\mathbf{r}$ and its first two derivatives with respect to s are continuous. Then, if P, P' are the points $\mathbf{r}$, $\mathbf{r}+\delta\mathbf{r}$ on the curve, corresponding to the values s and $s+\delta s$ respectively, the vector $\overrightarrow{PP'}$ is $\delta\mathbf{r}$.

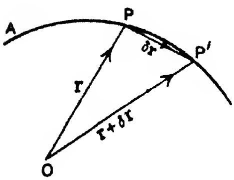

FIG. 47.

The quotient $\delta\mathbf{r}/\delta s$ is a vector in the same direction as $\delta\mathbf{r}$; and in the limit, as δs tends to zero, this direction becomes that of the tangent at P. And further, since the ratio of the lengths of the chord PP' and the arc PP' tends to unity as P' moves up to coincidence with P, the limiting value of the module of $\delta\mathbf{r}/\delta s$ is unity. Thus

$$\frac{d\mathbf{r}}{ds} = \underset{\delta s \to 0}{\mathrm{Lt}} \frac{\delta\mathbf{r}}{\delta s} = \mathbf{t} \text{ (say) } \qquad \text{.............................(1)}$$

is a unit vector in the direction of the tangent to the curve at P. When no misunderstanding is possible we shall refer to it briefly as the *unit tangent*

If x, y, z are the Cartesian coordinates of P referred to rectangular axes through O,

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

and

$$\mathbf{t} = \frac{d\mathbf{r}}{ds} = \frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j} + \frac{dz}{ds}\mathbf{k},$$

the direction cosines of $\mathbf{t}$ being therefore $\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$.

The vector equation of the tangent line at P may be easily written down. For the position vector $\mathbf{R}$ of any point on the tangent is given by

$$\mathbf{R}=\mathbf{r}+u\mathbf{t},$$

where u is a variable number, positive or negative. This is the equation of the tangent.

59. Curvature. Principal normal. The curvature of the curve at any point is the arc-rate of rotation of the tangent. Thus, if $\delta\theta$ is the circular measure of the angle between the tangents at P and P', $\frac{\delta\theta}{\delta s}$ is the average curvature of the arc PP'. The limiting value of this as δs tends to zero is the curvature at the point P. We shall denote it by κ. Thus

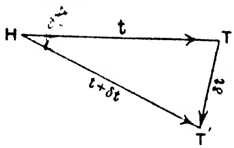

FIG. 48.

$$\kappa=\underset{\delta s\to 0}{\mathrm{Lt}}\frac{\delta\theta}{\delta s}=\frac{d\theta}{ds}.$$

Though $\mathbf{t}$ is a unit vector, it is a function of s, its direction changing from point to point of the curve. Let $\mathbf{t}$ be its value at P, and $\mathbf{t}+\delta\mathbf{t}$ its value at P', equal respectively to $\overrightarrow{HT}$ and $\overrightarrow{HT'}$. Then $\overrightarrow{TT'}$ is $\delta\mathbf{t}$, and the angle THT' is equal to the angle between the tangents at P and P', that is $\delta\theta$. The quotient $\delta\mathbf{t}/\delta s$ is a vector in the same direction as $\delta\mathbf{t}$, and in the limit as $\delta s\to 0$, this direction is perpendicular to the tangent at P. Since $\overrightarrow{HT}$ is a unit vector, the module of the limiting value of $\delta\mathbf{t}/\delta s$ is $\mathrm{Lt}\,\frac{\delta\theta}{\delta s}=\kappa$. Hence the relation

$$\frac{d\mathbf{t}}{ds}=\underset{\delta s\to 0}{\mathrm{Lt}}\frac{\delta\mathbf{t}}{\delta s}=\kappa\mathbf{n}, \quad\text{.....................................(2)}$$

where $\mathbf{n}$ is a unit vector perpendicular to the tangent at P, and in the plane of the tangents at P and a consecutive point P'. This plane, passing through three consecutive points at P, may be called the *plane of curvature* or the *local plane* of the curve at P. It is also commonly called the *osculating plane*.

The unit vectors $\mathbf{t}$, $\mathbf{n}$ are perpendicular to each other, and their plane is the plane of curvature. The straight line through P

parallel to $\mathbf{n}$ is called the *principal normal* at P. Its equation is clearly

$$\mathbf{R} = \mathbf{r} + u\mathbf{n},$$

where $\mathbf{R}$ is the position vector of any point on it. The curvature κ is a positive quantity; but the direction of rotation of the tangent is given by the vector $\mathbf{n}$, which we may call the *unit* (*principal*) *normal*. In virtue of (1), the equation (2) may also be written

$$\frac{d^2\mathbf{r}}{ds^2} = \kappa\mathbf{n}.$$

Squaring both sides, we have a well-known formula for the curvature

$$\kappa^2 = \left(\frac{d^2\mathbf{r}}{ds^2}\right)^2 = \left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2.$$

If $\mathbf{R}$ is any point in the plane of curvature, the vectors $\mathbf{R}-\mathbf{r}$, $\mathbf{t}$ and $\mathbf{n}$ are coplanar. Hence

$$[\mathbf{R}-\mathbf{r}, \mathbf{t}, \mathbf{n}] = 0,$$

and this is the equation of the osculating plane.

The *normal plane* at P is the plane through P perpendicular to the tangent. Hence its equation is

$$(\mathbf{R}-\mathbf{r})\cdot\mathbf{t} = 0.$$

The *circle of curvature* at P is the circle passing through three points on the curve ultimately coincident at P. Its radius is called the *radius of curvature*, and its centre *the centre of curvature*. This circle clearly lies in the osculating plane at P; and we leave it as an exercise for the student to show that the radius of curvature ρ is given by $\rho = \frac{1}{\kappa}$. The centre of curvature C then lies on the principal normal, so that

$$\overrightarrow{PC} = \rho\mathbf{n} = \frac{1}{\kappa}\mathbf{n}.$$

60. Binormal. Torsion. The straight line through P perpendicular to the plane of curvature is called the *binormal*. It is parallel to the vector $\mathbf{b} = \mathbf{t}\times\mathbf{n}$, and $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$ form a right-handed system of mutually perpendicular unit vectors. We may speak of $\mathbf{b}$ as the unit binormal.

Since $\mathbf{b}$ is a vector of constant length, it follows, as in Art. 56, that $\frac{d\mathbf{b}}{ds}$ is perpendicular to $\mathbf{b}$. Further, by differentiating the

relation $\mathbf{t}\cdot\mathbf{b}=0$, we find

$$\kappa\mathbf{n}\cdot\mathbf{b}+\mathbf{t}\cdot\frac{d\mathbf{b}}{ds}=0.$$

The first term is zero because $\mathbf{n}$ is perpendicular to $\mathbf{b}$, and the equation then shows that $\frac{d\mathbf{b}}{ds}$ is perpendicular to $\mathbf{t}$. But it is also

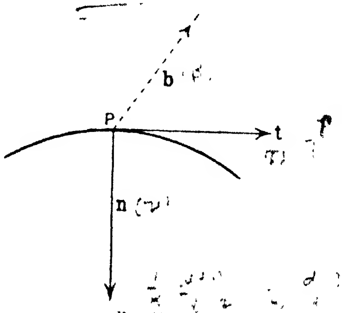

FIG 49.

perpendicular to $\mathbf{b}$, and must therefore be parallel to $\mathbf{n}$. We may then write

$$\frac{d\mathbf{b}}{ds}=-\lambda\mathbf{n}. \qquad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

And just as in formula (2) the scalar κ measures the arc-rate of turning of the unit vector $\mathbf{t}$, so here λ measures the arc-rate of turning of the unit vector $\mathbf{b}$. This rate of turning of the binormal is called the *torsion* of the curve at the point P. It is, of course, the arc-rate of rotation of the plane of curvature, since $\mathbf{b}$ is perpendicular to this plane. The negative sign in (3) indicates that the torsion is regarded as positive when the rotation of the binormal as s increases is right-handed relative to the vector $\mathbf{t}$. A glance at the figure shows that in this case $\frac{d\mathbf{b}}{ds}$ has the opposite direction to $\mathbf{n}$.

Having found the derivatives of $\mathbf{t}$ and $\mathbf{b}$, we can deduce that of $\mathbf{n}$. For

$$\frac{d\mathbf{n}}{ds}=\frac{d}{ds}\mathbf{b}\times\mathbf{t}=\mathbf{b}\times(\kappa\mathbf{n})-\lambda\mathbf{n}\times\mathbf{t}$$

$$=-\kappa\mathbf{t}+\lambda\mathbf{b}. \qquad \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

The value of the torsion may now be found in terms of the derivatives of $\mathbf{r}$ with respect to s. Since the second derivative

of $\mathbf{r}$ is $\kappa\mathbf{n}$, the third is $\frac{d}{ds}(\kappa\mathbf{n})$. Forming then the scalar triple product of the first three derivatives of $\mathbf{r}$, and neglecting those triple products which contain a repeated factor, we have

$$\left[\frac{d\mathbf{r}}{ds}\,\frac{d^2\mathbf{r}}{ds^2}\,\frac{d^3\mathbf{r}}{ds^3}\right]=\left[\mathbf{t},\ \kappa\mathbf{n},\ \kappa\frac{d\mathbf{n}}{ds}+\frac{d\kappa}{ds}\mathbf{n}\right]$$
$$=[\mathbf{t},\ \kappa\mathbf{n},\ \kappa(\lambda\mathbf{b}-\kappa\mathbf{t})],\quad \text{by (4)},$$
$$=\lambda\kappa^2[\mathbf{tnb}]=\lambda\kappa^2.$$

Hence the value of the torsion is given by

$$\lambda=\frac{1}{\kappa^2}\left[\frac{d\mathbf{r}}{ds}\,\frac{d^2\mathbf{r}}{ds^2}\,\frac{d^3\mathbf{r}}{ds^3}\right].$$

The *equation of the binormal* is

$$\mathbf{R}=\mathbf{r}+u\mathbf{b}.$$

Or, since $\mathbf{b}=\mathbf{t}\times\mathbf{n}$, this may also be put in the form

$$\mathbf{R}=\mathbf{r}+v\frac{d\mathbf{r}}{ds}\times\frac{d^2\mathbf{r}}{ds^2},$$

u, v being variable numbers.

Definite Integrals. Line and Surface Integrals.

61. Definite integral of a vector function. Let $\mathbf{f}$ be a given vector function of the scalar variable t, finite and continuous for values of t ranging from a to b. To indicate the functional dependence we may use the ordinary notation $\mathbf{f}(t)$ for the vector. Let the range $b-a$ be divided into a number of sub-ranges, which correspond to increments δt_1, δt_2, . . . , δt_n of the variable t. Also let $\mathbf{f}(t_1)$ be one of the values assumed by $\mathbf{f}$ in the first sub-range, $\mathbf{f}(t_2)$ one of the values assumed in the second, and so on. Consider the sum

$$\mathbf{S}=\delta t_1\mathbf{f}(t_1)+\delta t_2\mathbf{f}(t_2)+\ldots+\delta t_n\mathbf{f}(t_n).$$

It can be shown that, with the ordinary functions occurring in practical applications, if the number of sub-ranges increases indefinitely, and each of the increments δt tends to zero, this sum $\mathbf{S}$ tends to a definite finite limit which is independent of the mode of subdivision of the range $b-a$. This limiting value of $\mathbf{S}$ is equal to the difference of the values of the indefinite integral

$\mathbf{F}(t)$ of the function $\mathbf{f}(t)$ for the values b and a of the variable t; that is

$$\text{Lt}\ \mathbf{S} = \mathbf{F}(b) - \mathbf{F}(a).$$

This is called the *definite integral* of the function $\mathbf{f}(t)$ between the limits a and b, and is denoted by

$$\int_a^b \mathbf{f}(t)dt.$$

The result may be put in the form

$$\int_a^b \mathbf{f}(t)dt = \text{Lt} \sum_a^b \mathbf{f}(t) \,.\, \delta t = \mathbf{F}(b) - \mathbf{F}(a),$$

where $\mathbf{f}(t)$, in the second expression, is one of the values taken by $\mathbf{f}$ in the sub-range corresponding to the increment δt, the summation taking in all increments of t from a to b, and each of these increments tending to zero as a limit.

62. By way of illustration consider the displacement of a point P, moving with a variable velocity $\mathbf{v}$, which is a function of the time variable t. Let the interval from t_0 to t_1 be divided into a large number of infinitesimal intervals. During one of these, whose duration is δt, the displacement of the point is $\delta t \,.\, \mathbf{v}(t)$, where $\mathbf{v}(t)$ is a value assumed by $\mathbf{v}$ during this interval. The total displacement during the interval from t_0 to t_1 is the vector sum of the displacements during the sub-intervals. Taking the limiting value when each of the quantities δt tends to zero, we find for the total displacement of the point

$$\text{Lt} \sum_{t_0}^{t_1} \mathbf{v}(t) \,.\, \delta t = \int_{t_0}^{t_1} \mathbf{v}(t)dt.$$

Similarly, if the moving point has a variable acceleration $\mathbf{a}(t)$, the increment in the velocity during the short interval δt is $\delta t \,.\, \mathbf{a}(t)$; and the total increase in velocity during the interval t_0 to t_1 is

$$\text{Lt} \sum_{t_0}^{t_1} \mathbf{a}(t) \,.\, \delta t = \int_{t_0}^{t_1} \mathbf{a}(t)dt.$$

The position of the *centre of mass* of a body may be found by imagining the body divided up, in some convenient manner, into a large number of small portions, and proceeding to the limit when the number tends to infinity, and each element of volume converges to a point. If δv is the element of volume round the point P, whose position vector relative to a given origin

is $\mathbf{r}$, and μ the density at this point, the mass within the element of volume is $\mu\,.\,\delta v$. Then, since the c.m. of a system of particles is given by the formula
$$\bar{\mathbf{r}} = \Sigma m\mathbf{r}/\Sigma m,$$
the c.m. of the infinite number of particles, arrived at by proceeding to the limit as above, is given by
$$\mathbf{r} = \frac{\mathrm{Lt}\,\Sigma\mathbf{r}\mu\,.\,\delta v}{\mathrm{Lt}\,\Sigma\mu\,.\,\delta v} = \frac{1}{M}\int \mu\mathbf{r}\,dv,$$
where M is the mass of the whole body, and the range of integration includes the whole volume occupied by the body.

63.* Tangential line integral of a vector function. Consider a given curve, and a vector function $\mathbf{F}$ of the length s of the arc of this curve measured from a fixed point on it. Let A, B be two points on the curve, for which s has the values a and b respectively. If $\mathbf{t}$ is the unit tangent at a point of the curve, $\mathbf{F}\cdot\mathbf{t}$ is the measure of the resolute of $\mathbf{F}$ in the direction of the tangent. The definite integral of this quantity with respect to s, between the limits a and b, is called the line integral of the vector $\mathbf{F}$ along the curve from A to B. We write it
$$\int_a^b \mathbf{F}\cdot\mathbf{t}\,ds = \mathrm{Lt}\sum_a^b \mathbf{F}\cdot\mathbf{t}\,\delta s.$$
It is also frequently written
$$\int_A^B \mathbf{F}\cdot d\mathbf{r} = \mathrm{Lt}\sum_A^B \mathbf{F}\cdot\delta\mathbf{r},$$
where A, B are the end points of the arc of integration, and $\delta\mathbf{r}$ is the infinitesimal vector $\delta s\,.\,\mathbf{t}$ parallel to the tangent at the point considered.

If, for instance, the vector $\mathbf{F}$ represents the force acting on a particle which moves along the curve from A to B, $\mathbf{F}\cdot\delta\mathbf{r}$ measures the work done by the force during the infinitesimal displacement $\delta\mathbf{r}$; and the definite integral from A to B represents the total work done by the force during the displacement from A to B. If $\mathbf{F}$ represents the value of the electric (or magnetic) intensity at P, the line integral represents the work done on unit charge (or pole) as it moves from A to B; that is, the difference of potential between those two points. In hydrodynamics, if $\mathbf{v}$

represents the velocity of a particle of the fluid, the line integral

$$\int \mathbf{v}\cdot\mathbf{t}\, ds \equiv \int \mathbf{v}\cdot d\mathbf{r}$$

taken round any closed curve drawn in the fluid is called the *circulation* round the curve.

64.* Normal surface integral of a vector function. Consider a curved surface, and a vector $\mathbf{F}$ varying from point to point of the surface, and possessing at each point a definite value. Let $\mathbf{n}$ be a unit vector parallel to the normal at the point P of the surface, drawn outwards if the surface is closed, or always toward the same side if it is not closed. Then $\mathbf{F}\cdot\mathbf{n}$ is the resolved part of $\mathbf{F}$ along the normal. If the surface is divided up into a large number of small elements, and δA is the area of the element round P, the sum

$$S = \Sigma\mathbf{F}\cdot\mathbf{n}\,\delta A$$

extended to all the elements of the surface tends to a definite limiting value when each of the quantities δA tends to zero, and their number tends to infinity. This limiting value is called the *surface integral* of the function $\mathbf{F}$ over the given curved surface. We write it

$$\int \mathbf{F}\cdot\mathbf{n}\, dA = \mathrm{Lt}\,\Sigma\mathbf{F}\cdot\mathbf{n}\,\delta A.$$

It is identical with the surface integral of the scalar function $\mathbf{F}\cdot\mathbf{n}$; and, after the formation of this scalar product, is a matter of ordinary calculus.

The vector $\delta A\,.\,\mathbf{n}$ represents the vector area of the element of the surface. It is often written $\delta\mathbf{A}$, and the above equation put in the form

$$\int \mathbf{F}\cdot d\mathbf{A} = \mathrm{Lt}\,\Sigma\mathbf{F}\cdot\delta\mathbf{A}.$$

If, for instance, the vector $\mathbf{F}$ represents the value of the electric or magnetic induction at the point P, the surface integral gives the value of the total normal induction over the surface. Or if the surface is drawn in the region occupied by a liquid, whose velocity $\mathbf{v}$ varies from point to point, the surface integral of the vector $\mathbf{v}$ gives the rate at which liquid is flowing across the surface, in units of volume per unit time.

Note. Some worked examples will be found among the following exercises.

EXERCISES ON CHAPTER V.

1. Differentiate the following expressions, in which $\mathbf{r}$ is a function of t, r its module, and the other quantities are constants:

i. $r^2\mathbf{r}+\mathbf{a}\cdot\mathbf{r}\mathbf{b}$. ii. $r^3\mathbf{r}+\mathbf{a}\times\dfrac{d\mathbf{r}}{dt}$.

iii. $(a\mathbf{r}+r\mathbf{b})^2$. iv. $\dfrac{\mathbf{r}}{r^2}+\dfrac{r\mathbf{b}}{\mathbf{a}\cdot\mathbf{r}}$.

v. $\mathbf{r}^2+\dfrac{1}{\mathbf{r}^2}$. vi. $\frac{1}{2}m\left(\dfrac{d\mathbf{r}}{dt}\right)^2$.

2. Find the first and second derivatives of the products

$$\left[\mathbf{r}\ \frac{d\mathbf{r}}{dt}\ \frac{d^2\mathbf{r}}{dt^2}\right] \quad\text{and}\quad \mathbf{r}\times\left(\frac{d\mathbf{r}}{dt}\times\frac{d^2\mathbf{r}}{dt^2}\right).$$

3. Differentiate $\quad \dfrac{\mathbf{r}+\mathbf{a}}{\mathbf{r}^2+\mathbf{a}^2}\quad$ and $\quad\dfrac{\mathbf{r}\times\mathbf{a}}{\mathbf{r}\cdot\mathbf{a}}$.

4. If n, $\mathbf{a}$, $\mathbf{b}$ are constants, and

$$\mathbf{r}=\cos nt\,\mathbf{a}+\sin nt\,\mathbf{b},$$

prove that

$$\frac{d^2\mathbf{r}}{dt^2}+n^2\mathbf{r}=0,$$

and that

$$\mathbf{r}\times\frac{d\mathbf{r}}{dt}=n\mathbf{a}\times\mathbf{b}.$$

5. Find values of $\mathbf{r}$ satisfying the equations

i. $\dfrac{d^2\mathbf{r}}{dt^2}=\mathbf{a}$. ii. $\mathbf{a}\times\dfrac{d^2\mathbf{r}}{dt^2}=\mathbf{b}$,

where $\mathbf{a}$ and $\mathbf{b}$ are constant.

6. Find $\mathbf{r}$ from the equation $\dfrac{d^2\mathbf{r}}{dt^2}=\mathbf{a}t+\mathbf{b}$, given that both $\mathbf{r}$ and $\dfrac{d\mathbf{r}}{dt}$ vanish when $t=0$.

7. Interpret the relations $\mathbf{r}\cdot\dfrac{d\mathbf{r}}{ds}=0$ and $\mathbf{r}\times\dfrac{d\mathbf{r}}{ds}=0$.

8. i. State the condition that a given curve may be a plane curve.

ii. Put into Cartesian form the values of κ and λ, and the equations to the tangent, principal normal, binormal, plane of curvature and normal plane.

9. By means of the relation $\mathbf{t} = \frac{d}{ds}(r\hat{\mathbf{r}})$, prove that for any curve

$$\hat{\mathbf{r}}\cdot\mathbf{t} = \frac{dr}{ds}.$$

Hence, using the value $p = -\mathbf{r}\cdot\mathbf{n}$ of the length of the perpendicular from O to the tangent, prove that for a *plane* curve

$$\frac{dp}{ds} = \kappa r\frac{dr}{ds},$$

and therefore that

$$\kappa = \frac{1}{r}\frac{dp}{dr}.$$

10. The circular helix. This is a curve drawn on the surface of a right circular cylinder, and cutting the generators at a constant angle. Let a be the radius of the cylinder, $\frac{\pi}{2} - \alpha$ the angle at which the curve cuts the generators, and $\mathbf{k}$ the unit vector in the direction of the axis of the cylinder. Then, with a fixed point on the axis as origin, the position vector of any point on the helix may be expressed in the form

$$\mathbf{r} = a\cos\theta\,\mathbf{i} + a\sin\theta\,\mathbf{j} + a\theta\tan\alpha\,\mathbf{k}.$$

Differentiation with respect to s gives

$$\mathbf{t} = \frac{d\mathbf{r}}{ds} = a\frac{d\theta}{ds}(-\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j} + \tan\alpha\,\mathbf{k}).$$

Since this is a unit vector its square is unity, showing that

$$a^2\sec^2\alpha\left(\frac{d\theta}{ds}\right)^2 = 1.$$

Thus the derivative of θ is constant. Further, to find κ we have

$$\kappa\mathbf{n} = \frac{d^2\mathbf{r}}{ds^2} = -a\left(\frac{d\theta}{ds}\right)^2(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}),$$

which shows that the principal normal is always perpendicular to $\mathbf{k}$, and therefore to the axis of the cylinder. On squaring both sides of the last equation, we find

$$\kappa^2 = a^2\left(\frac{d\theta}{ds}\right)^4,$$

that is

$$\kappa = \frac{1}{a}\cos^2\alpha.$$

To find the torsion, we have

$$\frac{d^3\mathbf{r}}{ds^3} = \left(\frac{d\theta}{ds}\right)^3(a\sin\theta\,\mathbf{i} - a\cos\theta\,\mathbf{j}),$$

whence

$$\frac{d^2\mathbf{r}}{ds^2}\times\frac{d^3\mathbf{r}}{ds^3} = a^2\left(\frac{d\theta}{ds}\right)^5\mathbf{k}.$$

Therefore $$\kappa^2\lambda = \frac{d\mathbf{r}}{ds}\cdot\frac{d^2\mathbf{r}}{ds^2}\times\frac{d^3\mathbf{r}}{ds^3} = a^3\tan\alpha\left(\frac{d\theta}{ds}\right)^6.$$

Substituting the value of κ, we find

$$\lambda = \frac{1}{a}\sin\alpha\cos\alpha.$$

11. Prove that the locus of the centre of curvature of a circular helix is also a circular helix; and find the condition that it may be traced on the same cylinder.

The centre of curvature is the point

$$\mathbf{c} = \mathbf{r} + \frac{\mathbf{n}}{\kappa}.$$

Substitution of the values of these quantities gives the result.

12. Prove that the curvature of the curve defined by

$$\mathbf{r} = 2a\cos\theta\,\mathbf{i} + 2a\sin\theta\,\mathbf{j} + b\theta^2\mathbf{k}$$

is equal to $$\frac{a(a^2+b^2+b^2\theta^2)^{\frac{1}{2}}}{2(a^2+b^2\theta^2)^{\frac{3}{2}}}.$$

13. Find κ, and the equations to the principal normal and the plane of curvature, for the curve

$$\mathbf{r} = 4a\cos^3\theta\,\mathbf{i} + 4a\sin^3\theta\,\mathbf{j} + 3c\cos 2\theta\,\mathbf{k}.$$

14. In the curve

$$\mathbf{r} = a(3t - t^3)\mathbf{i} + 3at^2\mathbf{j} + a(3t + t^3)\mathbf{k},$$

show that $$\kappa = \lambda = \frac{1}{3a(1+t^2)^2}.$$

15. If the curve

$$\mathbf{r} = a\cos\theta\mathbf{i} + b\sin\theta\mathbf{j} + f(\theta)\mathbf{k}$$

is a plane curve, determine the form of $f(\theta)$.

16. Prove that for any curve $\frac{d\mathbf{t}}{ds}\cdot\frac{d\mathbf{b}}{ds} = -\kappa\lambda$.

17. If the tangent and the binormal at a point of a curve make angles θ, ϕ respectively with a fixed direction, show that

$$\frac{\sin\theta}{\sin\phi}\frac{d\theta}{d\phi} = -\frac{\kappa}{\lambda}.$$

18. A curve is drawn on a right circular cone, always inclined at the same angle β to the axis. Prove that

$$\lambda = \kappa\cot\beta.$$

19. A circle of radius a is drawn on a sheet of paper, which is then folded to form a cylinder of radius b. Show that for the new curve

$$\kappa^2 = \frac{1}{a^2} + \frac{1}{b^2}\cos^4\frac{s}{a},$$

where s is the length of the arc measured from a certain point.

20. If κ is the curvature of a curve, then that of its projection on a plane inclined at an angle β to the plane of curvature is $\kappa\cos\beta$ if the plane is parallel to the tangent, and $\kappa\sec^2\beta$ if it is parallel to the principal normal.

21. Prove that the circular helix is the only curve whose curvature and torsion are both constant.

From the relation $\frac{d^2\mathbf{r}}{ds^2} = \kappa\mathbf{n}$ it follows by differentiation, since κ is constant, that

$$\frac{d^3\mathbf{r}}{ds^3} = \kappa(-\kappa\mathbf{t} + \lambda\mathbf{b}),$$

and therefore

$$\frac{d^2\mathbf{r}}{ds^2} \times \frac{d^3\mathbf{r}}{ds^3} = \kappa^3\mathbf{b} + \kappa^2\lambda\mathbf{t},$$

and this is a constant vector, for its derivative is easily shown to vanish. Denote it by $\mathbf{d}$. Then

$$\mathbf{t}\cdot\mathbf{d} = \kappa^2\lambda$$

and

$$\mathbf{b}\cdot\mathbf{d} = \kappa^3,$$

while

$$\mathbf{n}\cdot\mathbf{d} = 0.$$

Thus the inclination of the tangent to $\mathbf{d}$ is constant, the principal normal is perpendicular to $\mathbf{d}$, and the plane of curvature has a constant inclination to a plane perpendicular to $\mathbf{d}$. From this last fact and the constancy of κ it follows, by the last exercise, that the curvature of the projection of the curve on a plane perpendicular to $\mathbf{d}$ is constant The theorem is then obvious.

22. Radius of spherical curvature. The sphere which passes through four points on a curve ultimately coincident with the point P is called the osculating sphere to the curve at that point. Its centre and radius are called the centre and radius of spherical curvature. Relative to an origin O let $\mathbf{c}$ be the position vector of the centre C of spherical curvature, and $\mathbf{r}$ that of the point P. Then the vector $\overrightarrow{PC} = \mathbf{c} - \mathbf{r}$ determines the radius of spherical

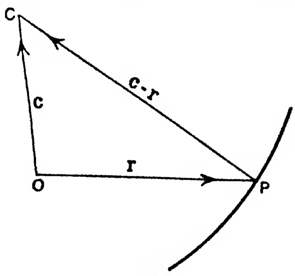

FIG. 50.

curvature. Let it be denoted by $\mathbf{R}$ and its module by R. Then, since the sphere and curve have four points in common, the first three derivatives of $\mathbf{c}$ and $\mathbf{R}^2$ with respect to s vanish. Hence, by differentiating

$$(\mathbf{c}-\mathbf{r})^2 = \mathbf{R}^2,$$

we have

$$(\mathbf{c}-\mathbf{r})\cdot\mathbf{t} = 0. \quad\text{.........(1)}$$

Another differentiation gives

$$(\mathbf{c}-\mathbf{r})\cdot\frac{d^2\mathbf{r}}{ds^2} = 1, \quad\text{.........(2)}$$

and a third leads to

$$(\mathbf{c}-\mathbf{r})\cdot\frac{d^3\mathbf{r}}{ds^3} = 0. \quad\text{.........(3)}$$

From (1) and (3) it follows that $\mathbf{R}$ is perpendicular to both $\frac{d\mathbf{r}}{ds}$ and $\frac{d^3\mathbf{r}}{ds^3}$, and is therefore of the form

$$\mathbf{R} = k\frac{d\mathbf{r}}{ds}\times\frac{d^3\mathbf{r}}{ds^3}.$$

To find the value of k multiply scalarly by $\frac{d^2\mathbf{r}}{ds^2}$. Then, in virtue of (2), we find

$$1 = -k\left[\frac{d\mathbf{r}}{ds}\ \frac{d^2\mathbf{r}}{ds^2}\ \frac{d^3\mathbf{r}}{ds^3}\right],$$

so that

$$\mathbf{R} = \frac{\frac{d^3\mathbf{r}}{ds^3}\times\frac{d\mathbf{r}}{ds}}{\left[\frac{d\mathbf{r}}{ds}\ \frac{d^2\mathbf{r}}{ds^2}\ \frac{d^3\mathbf{r}}{ds^3}\right]}. \quad\text{.........(4)}$$

This is the vector $\overrightarrow{PC}$. It may also be put in the form

$$\mathbf{R} = -\frac{1}{\kappa^2\lambda}\mathbf{t}\times\frac{d}{ds}(\kappa\mathbf{n})$$

$$= -\frac{1}{\kappa^2\lambda}\mathbf{t}\times\left(\frac{d\kappa}{ds}\mathbf{n}+\kappa\lambda\mathbf{b}-\kappa^2\mathbf{t}\right)$$

or

$$\mathbf{R} = \frac{1}{\kappa}\mathbf{n}+\frac{1}{\lambda}\frac{d}{ds}\left(\frac{1}{\kappa}\right)\mathbf{b} \quad\text{.........(5)}$$

This gives the well-known formula

$$R^2 = \frac{1}{\kappa^2}+\frac{1}{\lambda^2}\left(\frac{d}{ds}\frac{1}{\kappa}\right)^2 \quad\text{.........(6)}$$

23. Find the radius of spherical curvature for the circular helix.

24. Show that, for any curve,

$$\frac{1}{\kappa^4}\left(\frac{d^3\mathbf{r}}{ds^3}\right)^2 = 1+\lambda^2\mathbf{R}^2.$$

25. Find the radius of spherical curvature for the curve in Exercise **14**.

26. A variable point P on a given curve has a position vector $\mathbf{r}$ relative to a fixed origin O. Show that the vector area of the surface traced out by OP as P moves along the curve from A to B is given by

$$\frac{1}{2}\int_A^B \mathbf{r}\times\mathbf{t}\, ds = \frac{1}{2}\int_A^B \mathbf{r}\times d\mathbf{r}.$$

27. Show that, tor a closed surface,

$$\int \mathbf{n}\, dS = 0,$$

where $\mathbf{n}$ is the unit normal, and dS the area of an element of the surface. (The vector area of a closed surface is zero.)

CHAPTER VI.

KINEMATICS AND DYNAMICS OF A PARTICLE.

1. Kinematics.

65. Velocity at an instant. Let P be a moving point, and $\mathbf{r}$ its position vector relative to another point O, which may be thought of as either moving or stationary. The velocity of P relative to O is the rate of change of P's position relative to O, and is therefore represented in measure and direction by the rate of change of $\mathbf{r}$. If $\delta\mathbf{r}$ is the increment in $\mathbf{r}$ during an interval δt seconds, the quotient $\frac{\delta\mathbf{r}}{\delta t}$ is the average rate of change of $\mathbf{r}$ during this interval, and therefore represents the average velocity of P relative to O during the interval δt. The limiting value of the quotient as δt tends to zero is a vector which represents the velocity of P relative to O at that instant. Briefly then, we may say that

$$\mathbf{v} = \underset{\delta t \to 0}{\mathrm{Lt}} \frac{\delta\mathbf{r}}{\delta t} = \frac{d\mathbf{r}}{dt} \quad \text{.............(1)}$$

is the instantaneous velocity of P relative to O.

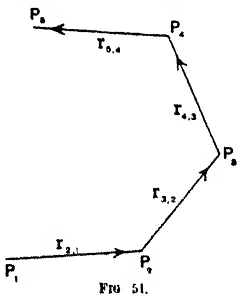

FIG. 51.

Suppose that there are n points P_1, P_2, ..., P_n, and that $\mathbf{r}_{m,\,m-1}$ is the position vector of the m^{th} relative to the $(m-1)^{\text{th}}$. Then that of the n^{th} relative to the first is

$$\mathbf{r}_{n,1} = \mathbf{r}_{n,\,n-1} + \mathbf{r}_{n-1,\,n-2} + \ldots + \mathbf{r}_{2,1}.$$

And since, as we have just seen, the derivative of the relative position vector is the relative velocity vector, differentiation of this equation gives

$$\mathbf{v}_{n,1} = \mathbf{v}_{n,\,n-1} + \mathbf{v}_{n-1,\,n-2} + \ldots + \mathbf{v}_{2,1}. \quad \text{.................(2)}$$

That is to say, the velocity of the n^{th} relative to the first is equal to the vector sum of the velocities of the n^{th} relative to the $(n-1)^{th}$, the $(n-1)^{th}$ relative to the $(n-2)^{th}$, and so on as far as that of the second relative to the first. This is the *theorem of vector addition of velocities.*

The case of three points is one of frequent occurrence, and the theorem may then be stated: the velocity of P_3 relative to P_2 is equal to the vector difference of the velocities of P_3 relative to P_1 and P_2 relative to P_1. This theorem was considered in Art. 13 for the case of uniform velocities only.

66. Acceleration at an instant. The acceleration of a point P relative to another, O, is the rate of change of the velocity of P relative to O. If $\mathbf{v}$ is the vector representing this velocity, then $\frac{d\mathbf{v}}{dt}$ represents the relative acceleration. In terms of their relative position vector $\mathbf{r} = \overrightarrow{OP}$, we may write

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2}$$

for the acceleration.

If there are n points it follows from the above that relative accelerations are compounded by vector addition. For differentiation of (2) gives

$$\mathbf{a}_{n,1} = \mathbf{a}_{n,n-1} + \mathbf{a}_{n-1,n-2} + \dots + \mathbf{a}_{2,1},$$

where the suffixes have the same meaning as above. This is the *theorem of vector addition of accelerations.*

67. Tangential and normal resolutes of acceleration. Consider the motion of a particle P along a fixed curve (Fig. 47). Let $\mathbf{r}$ be its position vector relative to a fixed origin O. Then, as already seen, the velocity of P at any instant is given by the vector $\mathbf{v} = \frac{d\mathbf{r}}{dt}$. But, by Art. 55, this may also be written

$$\mathbf{v} = \frac{d\mathbf{r}}{ds}\frac{ds}{dt} = v\mathbf{t}, \quad \dots\dots(1)$$

where $v = \frac{ds}{dt}$ is the speed of the particle along the path, and $\mathbf{t}$ is the unit tangent to the curve at that point. This equation merely expresses that the velocity has the direction of the tangent and the magnitude of the speed.

The vector representing the acceleration at the instant considered is

$$\mathbf{a}=\frac{d\mathbf{v}}{dt}=\frac{d}{dt}(v\mathbf{t})$$

$$=\frac{dv}{dt}\mathbf{t}+v\frac{d\mathbf{t}}{ds}\frac{ds}{dt}$$

or

$$\mathbf{a}=\frac{dv}{dt}\mathbf{t}+\kappa v^2\mathbf{n}, \qquad\qquad (2)$$

where κ is the curvature and $\mathbf{n}$ the unit principal normal. This formula shows that the acceleration of the particle is parallel to the plane of curvature. Its tangential resolute is measured by $\frac{dv}{dt}$, and its resolute parallel to the principal normal by κv^2. The former is the rate of increase of the speed and is independent of the shape of the curve. The latter depends on the curvature and the speed. If $\delta\theta$ is the angle turned through by the tangent at P during the interval δt, $\frac{d\theta}{dt}$ is the rate of rotation of the tangent. Denoting this by ω, we may write the normal resolute of the acceleration

$$v\frac{d\mathbf{t}}{d\theta}\frac{d\theta}{dt}=v\omega\mathbf{n}.$$

For uniform motion in a circular path of radius a the tangential resolute is zero. Thus the acceleration is always normal to the path and equal to v^2/a.

Note. Differentiations with respect to the time variable t are frequently denoted by placing dots over the quantity differentiated. Thus

$$\dot{v}=\frac{dv}{dt}, \quad \dot{\mathbf{r}}=\frac{d\mathbf{r}}{dt}, \quad \ddot{\mathbf{r}}=\frac{d^2\mathbf{r}}{dt^2},$$

and so on.

68. Radial and transverse resolutes of velocity and acceleration. Suppose now that the path of the moving particle P is a plane curve, and that it is required to find the resolutes of the velocity and acceleration parallel and perpendicular to $\mathbf{r}=\overrightarrow{OP}$. Let $\delta\theta$ be (the circular measure of) the angle turned through by OP during the interval δt, reckoned positive when the rotation is anti-clockwise. Then $\omega=\frac{d\theta}{dt}$ is the rate of turning of OP, or the angular velocity of P about O. Let $\hat{\mathbf{r}}$, $\hat{\mathbf{s}}$ be unit vectors

parallel and perpendicular to $\mathbf{r}$, the latter in the positive direction of ω. Then the points whose position vectors are $\hat{\mathbf{r}}$ and $\hat{\mathbf{s}}$, move in circles of unit radius with equal speeds ω. The velocity of the former is in the direction of $\hat{\mathbf{s}}$, and that of the latter in the direction of $-\hat{\mathbf{r}}$. Therefore $\dfrac{d\hat{\mathbf{r}}}{dt}=\omega\hat{\mathbf{s}}$; $\dfrac{d\hat{\mathbf{s}}}{dt}=-\omega\hat{\mathbf{r}}$.

If now r is the module of $\mathbf{r}$, the velocity of the point P is given by

$$\mathbf{v}=\frac{d\mathbf{r}}{dt}=\frac{d}{dt}(r\hat{\mathbf{r}})$$

$$=\dot{r}\hat{\mathbf{r}}+r\omega\hat{\mathbf{s}}, \quad \ldots\ldots\ldots\ldots(1)$$

showing that the radial and transverse resolutes of the velocity are $\dfrac{dr}{dt}$ and $r\dfrac{d\theta}{dt}$ respectively.

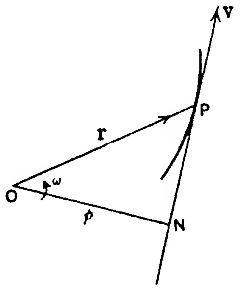

FIG. 52.

Similarly the acceleration of P is given by

$$\mathbf{a}=\frac{d\mathbf{v}}{dt}=\frac{d}{dt}(\dot{r}\hat{\mathbf{r}}+r\omega\hat{\mathbf{s}})$$

$$=(\ddot{r}\hat{\mathbf{r}}+\dot{r}\omega\hat{\mathbf{s}})+(\dot{r}\omega\hat{\mathbf{s}}+r\dot{\omega}\hat{\mathbf{s}}-r\omega^2\hat{\mathbf{r}})$$

$$=(\ddot{r}-r\dot{\theta}^2)\hat{\mathbf{r}}+(r\ddot{\theta}+2\dot{r}\dot{\theta})\hat{\mathbf{s}}. \quad \ldots\ldots\ldots\ldots(2)$$

The radial and transverse resolutes of the acceleration are therefore given by

$$\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2 \quad \text{and} \quad r\frac{d^2\theta}{dt^2}+2\frac{dr}{dt}\frac{d\theta}{dt}$$

respectively.

Exercises.

(1) *A particle P moves in a plane with constant angular velocity ω about O. If the rate of increase of its acceleration is parallel to PO, prove that*

$$\frac{d^2r}{dt^2} = \tfrac{1}{3}r\omega^2.$$

By formula (2) of the above Art., since ω is constant the acceleration of P is

$$\mathbf{a} = (\ddot{r} - r\omega^2)\hat{\mathbf{r}} + 2\dot{r}\omega\hat{\mathbf{s}}.$$

Since the rate of increase of this is parallel to $\hat{\mathbf{r}}$, the $\hat{\mathbf{s}}$-component of $\dot{\mathbf{a}}$ is zero. This is easily shown to be

$$3\ddot{r}\omega - r\omega^3 = 0,$$

so that

$$\ddot{r} = \tfrac{1}{3}r\omega^2$$

as required.

(2) *A particle P is moving on the surface of a body B with velocity $\mathbf{v}$ relative to the body, while the latter is rotating relative to surrounding objects with an angular velocity ω about a fixed axis parallel to $\hat{\mathbf{a}}$. Find the velocity of P relative to surrounding objects.*

With any point O on the fixed axis as origin, let $\mathbf{r}$ be the position vector of the particle P. Then the velocity of the point Q of the body which is instantaneously coincident with P is

$$\mathbf{u} = \omega\hat{\mathbf{a}}\times\mathbf{r},$$

since $\omega\hat{\mathbf{a}}$ is the angular velocity of the body. And the resultant velocity of P is the vector sum of its velocity relative to Q and the velocity of Q; that is

$$\mathbf{v} + \omega\hat{\mathbf{a}}\times\mathbf{r}.$$

69. Areal Velocity. Considering again the general case of motion of P, when its path is not necessarily a plane curve, we may form the moment about O of the velocity vector $\mathbf{v}$ regarded as localised in a line through P. As seen in Art. 40 this moment is the vector $\mathbf{r}\times\mathbf{v}$ perpendicular to the plane of $\mathbf{r}$ and $\mathbf{v}$. If $\mathbf{k}$ is the unit vector in this direction, and p the length of the perpendicular ON to the tangent at P, we may write

$$\mathbf{r}\times\mathbf{v} = pv\mathbf{k}.$$

But, using the value of $\mathbf{v}$ found in Art. 68, the particle moving instantaneously in the plane OPN, we have

$$\mathbf{r}\times\mathbf{v} = \mathbf{r}\times(\dot{r}\hat{\mathbf{r}} + r\omega\hat{\mathbf{s}}) = r^2\omega\mathbf{k},$$

where ω is the rate of turning of OP. Equating these two values for the moment, we find the relation

$$r^2\omega = pv. \quad \text{.......................................(1)}$$

The *areal velocity* of the point P about O is the rate of description of vector area by the line OP. This is represented by the vector $\frac{1}{2}\mathbf{r}\times\mathbf{v}$. For, in a short interval δt the displacement of P is $\delta t\mathbf{v}$, and the vector area swept out by OP is $\frac{1}{2}\mathbf{r}\times(\delta t\mathbf{v})$. The rate of description of vector area is therefore $\frac{1}{2}\mathbf{r}\times\mathbf{v}$. The measure of the areal velocity will be denoted by $\frac{1}{2}h$, so that

$$h = pv = r^2\omega. \quad \text{...................................(2)}$$

70. Motion with constant acceleration. This is a case of considerable importance, illustrated approximately by the motion of a projectile under gravity. Let $\mathbf{d}$ be a unit vector in the direction of the constant acceleration whose measure is g, and $\mathbf{r}$ the position vector of the moving particle P. Then

$$\frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} = g\mathbf{d}.$$

Integration with respect to t gives

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = gt\mathbf{d} + \mathbf{v}_0,$$

where $\mathbf{v}_0$ is the constant of integration, obviously representing the velocity at the instant $t=0$. Integrating again, we find for the position vector of P,

$$\mathbf{r} = \frac{1}{2}gt^2\mathbf{d} + t\mathbf{v}_0,$$

where we have made the constant of integration zero by choosing as origin the initial position of P. This equation shows that the locus of P is a parabola whose axis is in the direction of $\mathbf{d}$. For $\mathbf{r}$ is the sum of two vectors in fixed directions, with modules proportional to t^2 and t respectively.

2. Dynamics of a Particle.

71. Momentum. The momentum of a moving particle is a vector quantity, jointly proportional to the mass of the particle and to its velocity, and having the same direction as the velocity. The unit of momentum is chosen as that of a particle of unit mass moving with unit velocity. Hence, if m is the measure of the

mass, and $\mathbf{v}$, $\mathbf{M}$ vectors representing the velocity and moment of the particle respectively,

$$\mathbf{M} = m\mathbf{v}.$$

The rate of change of the momentum of the particle is als vector quantity, represented by

$$\frac{d\mathbf{M}}{dt} = m\frac{d\mathbf{v}}{dt} + \frac{dm}{dt}\mathbf{v}.$$

If, as is usually the case, the mass of the particle is constant,

$$\frac{d\mathbf{M}}{dt} = m\frac{d\mathbf{v}}{dt} = m\mathbf{a},$$

and is therefore in the same direction as the acceleration.

72. Newton's Second Law of Motion. According to this la which is the foundation of the ordinary theory of dynam the force acting on a particle is proportional to the rate of cha of momentum produced by it, and has the same direction. Hen with the choice of unit force as that which produces unit accele tion in a particle of unit mass, we have the relation

$$\mathbf{F} = \frac{d}{dt}(m\mathbf{v}),$$

where $\mathbf{F}$ is the vector representing the force. When m is consta this may be written simply

$$\mathbf{F} = m\mathbf{a},$$

showing that the acceleration produced in the motion of a parti of constant mass has the same direction as the force produci it. This equation is called the *equation of motion* for t particle.

Suppose that the particle is acted on by several forces $\mathbf{F}_1$, $\mathbf{F}_2, \ldots, \mathbf{F}_n$, which, if acting separately, would produce accele tions $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ respectively. Their joint effect on t particle is the same as that of a single force $\Sigma\mathbf{F}$ which produ an acceleration $\mathbf{a}'$ given by

$$m\mathbf{a}' = \sum_1^n \mathbf{F} = \sum_1^n m\mathbf{a} = m(\mathbf{a}_1 + \mathbf{a}_2 + \ldots + \mathbf{a}_n).$$

Thus the actual acceleration $\mathbf{a}'$ is equal to the vector sum the accelerations $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$. That is to say, the effect each force is uninfluenced by the action of the others. Ea

produces the same effect as if it were the only force acting on the particle.

Let $\mathbf{r}$ be the position vector of the particle relative to a fixed point O, and suppose that referred to rectangular axes through O,

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k},$$

$$\mathbf{F} = X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k}.$$

Then the equation of motion $m\mathbf{a} = \Sigma\mathbf{F}$ may be written

$$m\left(\frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j} + \frac{d^2z}{dt^2}\mathbf{k}\right) = \Sigma(X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k}).$$

This is equivalent to the three scalar equations

$$m\ddot{x} = \Sigma X, \quad m\ddot{y} = \Sigma Y, \quad m\ddot{z} = \Sigma Z,$$

which are the ordinary Cartesian equations of motion for the particle.

73. Impulse of a force. The impulse of a force $\mathbf{F}$ acting on a particle during any interval of time, is the change of momentum produced by it during that interval. It is therefore a vector quantity. If, during the interval, the velocity changes from $\mathbf{v}_0$ to $\mathbf{v}_1$ the impulse of the force is given by

$$\mathbf{I} = m(\mathbf{v}_1 - \mathbf{v}_0).$$

When the force is variable, and acts from the instant t_0 to the instant t_1, the definite integral

$$\int_{t_0}^{t_1} \mathbf{F}\,dt = m\int_{t_0}^{t_1} \mathbf{a}\,dt = m(\mathbf{v}_1 - \mathbf{v}_0),$$

by Art. 62, and therefore represents the impulse of the force. When the force is constant the value of the definite integral is simply $(t_1 - t_0)\mathbf{F}$. Briefly we say that the vector $\mathbf{I}$ is the time-integral of the vector $\mathbf{F}$.

An *impulsive force*, also frequently called an *impulse*, is a very large force acting for a very short time. Hence during the action of the force the position of the particle is practically unchanged. The effect of the impulsive force is represented completely by the change of momentum produced by it. We therefore specify an impulsive force by its impulse in the original sense.

74. Activity of a force. In accordance with the definition of work given in Art. 39, the work done by a force $\mathbf{F}$ acting on a

particle, during a small displacement $\delta\mathbf{r}$ of the particle, is measured by $\mathbf{F}\cdot\delta\mathbf{r}$. If this displacement takes place during an interval δt, the average *rate of working* during that interval is $\mathbf{F}\cdot\dfrac{\delta\mathbf{r}}{\delta t}$; and proceeding to the limit as $\delta t\to 0$, we find for the instantaneous rate of working, or the *activity* of the force,

$$\mathbf{F}\cdot\frac{d\mathbf{r}}{dt}=\mathbf{F}\cdot\mathbf{v}.$$

The work done during a short interval δt is $\mathbf{F}\cdot\mathbf{v}\,\delta t$; and the *total work* done by the force from the instant t_0 to the instant t_1 is therefore

$$\text{Lt}\,\Sigma\mathbf{F}\cdot\mathbf{v}\,\delta t=\int_{t_0}^{t_1}\mathbf{F}\cdot\mathbf{v}\,dt.$$

75. The principle of energy. The kinetic energy of a moving particle is a scalar quantity jointly proportional to its mass and to the square of its speed. The unit of kinetic energy is taken as twice that of a particle of unit mass moving with unit speed. Hence the kinetic energy of a particle of mass m moving with a velocity $\mathbf{v}$ is

$$T=\tfrac{1}{2}mv^2=\tfrac{1}{2}m\mathbf{v}^2.$$

If the velocity is variable, owing to the action of forces whose resultant is $\mathbf{F}$, the rate of increase of the kinetic energy is

$$\frac{dT}{dt}=m\mathbf{v}\cdot\frac{d\mathbf{v}}{dt}=m\mathbf{a}\cdot\mathbf{v}=\mathbf{F}\cdot\mathbf{v},$$

and is therefore equal to the activity of the resultant force $\mathbf{F}$. In other words, the rate of increase of the kinetic energy is equal to the rate at which the force $\mathbf{F}$ is doing work on the particle. It follows that, during any finite interval, the increase in the kinetic energy of the particle is equal to the work done during that interval by the resultant force acting on the particle. This is true whether $\mathbf{F}$ is constant or variable, and is called the *principle of energy* for the particle.

[If we begin by defining the kinetic energy of the particle as the work it can do in virtue of its velocity, we may find its value thus. Suppose that at the given instant t_0 the particle has a velocity $\mathbf{v}_0$ and that it does work against a variable force $\mathbf{F}$ till it is finally brought to rest at the instant t_1. Then at any instant the rate at which it is doing work against the force

$\mathbf{F}$ is $-\mathbf{F}\cdot\mathbf{v} = -m\dfrac{d\mathbf{v}}{dt}\cdot\mathbf{v}$; and therefore the total work done against the force before the particle is brought to rest is

$$-\int_{t_0}^{t_1} m\frac{d\mathbf{v}}{dt}\cdot\mathbf{v}\,dt = -\tfrac{1}{2}m[\mathbf{v}^2]_{t_0}^{t_1} = \tfrac{1}{2}m\mathbf{v}_0^2.$$

This then is the value of the kinetic energy due to the velocity $\mathbf{v}_0$.]

Suppose that, owing to the action of the force $\mathbf{F}$ from the instant t_0 to the instant t_1, the velocity of the particle changes from $\mathbf{v}_0$ to $\mathbf{v}_1$. The impulse of the force is the change of momentum; i.e.

$$\mathbf{I} = m(\mathbf{v}_1 - \mathbf{v}_0).$$

The increase in the kinetic energy during this interval is

$$\begin{aligned}\tfrac{1}{2}m\mathbf{v}_1^2 - \tfrac{1}{2}m\mathbf{v}_0^2 &= \tfrac{1}{2}m(\mathbf{v}_1 - \mathbf{v}_0)\cdot(\mathbf{v}_1 + \mathbf{v}_0)\\ &= \tfrac{1}{2}\mathbf{I}\cdot(\mathbf{v}_1 + \mathbf{v}_0),\end{aligned}$$

that is, half the scalar product of the impulse and the sum of the initial and final velocities.

76. Moment of momentum. Let $\mathbf{r}$ be the position vector of the moving particle relative to a fixed point O. Its velocity is $\mathbf{v} = \dot{\mathbf{r}}$ and its momentum $m\mathbf{v}$. Regarding the momentum as localised in a straight line through the particle, we have for its moment about O,

$$\mathbf{H} = \mathbf{r}\times m\mathbf{v}.$$

This *moment of momentum* of the particle about O is also called its **angular momentum** about O.

The rate of increase of this angular momentum is

$$\begin{aligned}\frac{d\mathbf{H}}{dt} &= \frac{d}{dt}(\mathbf{r}\times m\mathbf{v}) = \mathbf{v}\times m\mathbf{v} + \mathbf{r}\times m\frac{d\mathbf{v}}{dt}\\ &= \mathbf{r}\times\mathbf{F},\end{aligned}$$

where $\mathbf{F} = m\dot{\mathbf{v}}$ is the resultant force acting on the particle. Thus *the rate of increase of the angular momentum (A.M.) of the particle about O is equal* * *to the moment about O of the resultant force on the particle. This is the principle of angular momentum.*

In particular, if the resultant force has zero moment about O, the A.M. of the particle about that point remains constant. This is the principle of the *conservation of A.M.* for the particle.

* The term *equivectorial* may be used of vector quantities which have the same measure and direction, and are therefore represented by equal vectors. But such quantities are commonly said to be *equal*.

77. Central forces. If, for instance, the particle is acted on by a force always directed toward the point O, the moment of the force about O is zero, and therefore the A.M. of the particle about that point is constant. Assuming the mass of the particle invariable, we have the result that $\mathbf{r}\times\mathbf{v}$ is a constant vector perpendicular to the plane of $\mathbf{r}$ and $\mathbf{v}$. Hence the plane of $\mathbf{r}$ and $\mathbf{v}$ is invariable; that is to say, the particle moves in a *plane* curve, whose plane contains the point O. The path of the particle is called its orbit.

Such a force, directed always toward a fixed point, is called a *central force*, and the fixed point is the *centre of force.*

78. Central force varying inversely as the square of the distance. To find the path described by a particle acted on by a central force toward O, varying inversely as the square of the distance of the particle from O. Let $\mathbf{r}$ be the position vector of the particle relative to O, and $\mu m/r^2$ the measure of the force on the particle. The constant μ is called the *intensity of force*, representing the force per unit mass at unit distance from O. The acceleration of the particle has the direction of $-\mathbf{r}$, and is thus specified by the relation

$$\frac{d^2\mathbf{r}}{dt^2} = -\frac{\mu}{r^2}\hat{\mathbf{r}}. \qquad \text{.................................}(1)$$

Further, considering the areal velocity of the particle about O, we have (Art. 69)

$$\mathbf{r}\times\mathbf{v} = h\mathbf{k} = r^2\omega\mathbf{k},$$

where $\mathbf{k}$ is a unit vector perpendicular to the plane of the orbit, and ω the rate of turning of $\mathbf{r}$. From these equations it follows that

$$\frac{1}{\mu}\frac{d^2\mathbf{r}}{dt^2}\times(h\mathbf{k}) = -\frac{\hat{\mathbf{r}}}{r^2}\times(r^2\omega\mathbf{k}) = -\omega\hat{\mathbf{r}}\times\mathbf{k} = \omega\hat{\mathbf{s}},$$

where $\hat{\mathbf{s}}$ is the unit vector $-\hat{\mathbf{r}}\times\mathbf{k}$ in the plane of the orbit and perpendicular to $\hat{\mathbf{r}}$. But, by Art. 68, $\omega\hat{\mathbf{s}} = \dfrac{d\hat{\mathbf{r}}}{dt}$; and since, in the case of a central force, h is constant, we may write the last equation

$$\frac{1}{\mu}\frac{d}{dt}\left(\frac{d\mathbf{r}}{dt}\times h\mathbf{k}\right) = \frac{d\hat{\mathbf{r}}}{dt},$$

which on integration gives immediately

$$\frac{1}{\mu}\mathbf{v}\times h\mathbf{k} = \hat{\mathbf{r}} + e\hat{\mathbf{a}}, \qquad \text{.................}(2)$$

where $e\hat{\mathbf{a}}$ is the constant vector of integration, whose module is e. If now θ is the variable angle between $\mathbf{r}$ and $\hat{\mathbf{a}}$, the scalar product

$$\mathbf{r}\cdot\hat{\mathbf{a}} = r\cos\theta.$$

Hence, on multiplying the last equation scalarly by $\mathbf{r}$, and writing the second member first, we find

$$r(1 + e\cos\theta) = \frac{1}{\mu}\mathbf{r}\cdot\mathbf{v}\times h\mathbf{k} = \frac{1}{\mu}\mathbf{r}\times\mathbf{v}\cdot h\mathbf{k}$$

$$= \frac{h\mathbf{k}\cdot h\mathbf{k}}{\mu} = \frac{h^2}{\mu} = l \text{ (say).}$$

Hence the equation of the orbit of the particle is

$$\frac{l}{r} = 1 + e\cos\theta, \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

representing a conic whose eccentricity is e, whose semi-latus rectum is $l = h^2/\mu$, and one of whose foci is at the centre of force.

The orbit will be an ellipse, a parabola or an hyperbola according as e is less than, equal to or greater than unity. Squaring the value of $e\hat{\mathbf{a}}$ given by (2), and noticing that $\mathbf{v}$ is perpendicular to $\mathbf{k}$, we find

$$e^2 = \frac{h^2v^2}{\mu^2} - \frac{2h}{\mu}\hat{\mathbf{r}}\cdot\mathbf{v}\times\mathbf{k} + 1.$$

And since $\quad \hat{\mathbf{r}}\cdot\mathbf{v}\times\mathbf{k} = \mathbf{r}\times\mathbf{v}\cdot\mathbf{k}/r = h/r,$

it follows that $e^2 <, = \text{ or } > 1$ according as $v^2 <, = \text{ or } > \frac{2\mu}{r}$. If then the particle is projected with a speed V at a distance c from the centre of force, the orbit will be an ellipse, a parabola or an hyperbola according as V^2 is less than, equal to or greater than $2\mu/c$. In any case the value of the eccentricity is given by

$$e^2 = \frac{h^2}{\mu}\left(\frac{V^2}{\mu} - \frac{2}{c}\right) + 1. \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

The first case is one of considerable importance, being illustrated by the motion of the planets relative to the sun. The orbit of a planet relative to the sun is an ellipse with the sun at one of its foci. This fact, as well as the constancy of the areal velocity of a planet about the sun, and a relation between the size of a planet's orbit and its period of revolution, were first discovered by the astronomer Kepler, from whose observations

Newton deduced the inverse square law of gravitational attraction between any two bodies. (Cf. Exercise 18 at the end of this chapter.)

79. Planetary motion. On account of the importance attaching to the case of the motion of a particle in an elliptic orbit under a centre of force at one focus, we shall examine it a little closer. Let a, b be the semi-axes of the ellipse, whose area is then πab. The rate of description of area by the line joining the particle to the centre of force is $h/2$. Hence the *periodic time*, or the time of one complete revolution of the particle, is

$$T = \frac{\pi ab}{\frac{1}{2}h} = 2\pi ab \div \sqrt{\mu\frac{b^2}{a}} = \frac{2\pi}{\sqrt{\mu}}a^{\frac{3}{2}}, \quad \text{..................(5)}$$

since the semi-latus rectum is $l = b^2/a$. Hence for particles moving in different ellipses about the same centre of force, and with the same intensity (μ) of force, the squares of their periodic times are proportional to the cubes of the major axes of their orbits, This relation was observed by Kepler in the case of the planets.

To find the speed v of the particle at any point of its path, multiply (1) scalarly by $2\frac{d\mathbf{r}}{dt}$. Then

$$2\frac{d\mathbf{r}}{dt}\cdot\frac{d^2\mathbf{r}}{dt^2} = -\frac{2\mu}{r^3}\mathbf{r}\cdot\frac{d\mathbf{r}}{dt} = -\frac{2\mu}{r^2}\frac{dr}{dt}.$$

On integration, therefore,

$$v^2 = \left(\frac{d\mathbf{r}}{dt}\right)^2 = C + \frac{2\mu}{r}.$$

To find the constant C of integration, consider the speed v_1 at the end of the minor axis. The perpendicular distance to the tangent at this point is b, while the distance r from the focus is a. For this point the last equation becomes

$$C + \frac{2\mu}{a} = {v_1}^2 = \frac{h^2}{b^2} = \frac{\mu l}{b^2} = \frac{\mu}{a},$$

so that $C = -\mu/a$, and the speed at any point is given by

$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right). \quad \text{...........................(6)}$$

Another expression for the speed is sometimes useful. If p, p' are the perpendicular distances from the centre of force and the

other focus respectively, to the tangent to the ellipse at P, the speed at P is

$$v=\frac{h}{p}=\frac{hp'}{pp'}=\frac{hp'}{b^2}, \qquad\qquad (7)$$

and is thus proportional to p'.

Given the initial position of the particle and its velocity of projection, the orbit is determined. For the semi-major axis may be found at once from the equation (6), using the initial values of r and v. The other focus O' is on the line through the point P of projection, equally inclined with PO to the direction of projection; and since $OP+PO'=2a$, the point O' is known. The centre of the ellipse bisects OO', and the major axis lies along it.

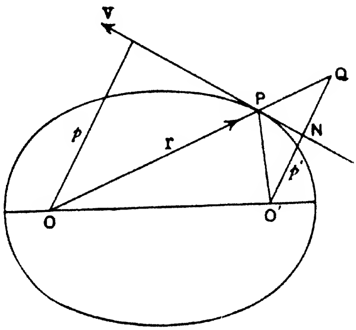

FIG 53.

The value $h=vp$ is found from initial values, and the semi-minor axis from the relation $h^2=\mu b^2/a$.

Examples.

(1) *A particle P describes an ellipse under a central force to the focus O. Show that its velocity at any instant may be resolved into two components of constant magnitude, perpendicular to the major axis and OP respectively.*

Let $O'N$ be the perpendicular from the other focus to the tangent at P. Then, by formula (7) above, the velocity $\mathbf{v}$ of the particle is proportional to $\overrightarrow{O'N}$ and at right angles to it. But if $O'N$ is produced to meet OP in Q,

$$\overrightarrow{O'N}=\tfrac{1}{2}\overrightarrow{O'Q}=\tfrac{1}{2}(\overrightarrow{O'O}+\overrightarrow{OQ}),$$

and both $O'O$ and OQ are of constant length. Hence the result, which is otherwise obvious from equation (2) of Art. 78.

(2) *A particle of mass m is moving in an ellipse with a focus as centre of force. At the end of the minor axis, in its motion from the centre of force, it receives a blow which changes its orbit to a circle. Find the blow.*

Let O be the focus, C the centre of the ellipse, and B' the end of the minor axis. Take $\mathbf{i}$, $\mathbf{j}$ unit vectors in the directions OC and $B'C$ respectively. Then, in virtue of formula (6), the original velocity at B' is

$$\mathbf{v}_1 = \sqrt{\frac{\mu}{a}}\,\mathbf{i}.$$

In the circular orbit the velocity must be perpendicular to OB' and the speed V such that

$$\frac{V^2}{a} = \frac{\mu}{a^2},$$

or $V = \sqrt{\mu/a}$, so that the speed is unaltered. Now, since

$$\overrightarrow{OB'} = ae\mathbf{i} - b\mathbf{j},$$

the unit vector at right angles to this, in the direction of the velocity $\mathbf{v}_2$ in the circular orbit, is

$$\frac{1}{a}(b\mathbf{i} + ae\mathbf{j}),$$

so that

$$\mathbf{v}_2 = \sqrt{\frac{\mu}{a}}\left(\frac{b}{a}\mathbf{i} + e\mathbf{j}\right).$$

Hence the blow is given by

$$m(\mathbf{v}_2 - \mathbf{v}_1) = m\sqrt{\frac{\mu}{a}}\left(e\mathbf{j} - \frac{a-b}{a}\mathbf{i}\right).$$

The module of this is $\dfrac{m}{a}\sqrt{2\mu(a-b)}$,

and its direction makes an angle

$$\tan^{-1}\sqrt{\frac{a+b}{a-b}}$$

with CO.

80.* Central force varying directly as the distance. Let the force on the particle toward O be $m\mu r$, where μ is the intensity of force and r the distance of the particle from O. Then the acceleration is $-\mu\mathbf{r}$, and the equation of motion for the particle is

$$\frac{d^2\mathbf{r}}{dt^2} = -\mu\mathbf{r}. \quad\text{.....................................}(1)$$

Multiplying scalarly by $2\dfrac{d\mathbf{r}}{dt}$ we have

$$2\frac{d\mathbf{r}}{dt}\cdot\frac{d^2\mathbf{r}}{dt^2} = -2\mu\mathbf{r}\cdot\frac{d\mathbf{r}}{dt},$$

which, on integration, gives

$$\left(\frac{d\mathbf{r}}{dt}\right)^2 = C - \mu\mathbf{r}^2.$$

If p is the perpendicular distance of O from the tangent at $\mathbf{r}$ to the orbit, this equation gives

$$\frac{1}{p^2} = \frac{v^2}{h^2} = \frac{1}{h^2}(C - \mu r^2).$$

But the p, r equation of an ellipse referred to its centre is

$$\frac{1}{p^2} = \frac{a^2 + b^2 - r^2}{a^2b^2},$$

where a, b are the semi-axes. Hence the orbit of the particle is an ellipse whose centre is O, while

$$h = \sqrt{\mu} \,.\, ab$$

and

$$C = \mu(a^2 + b^2).$$

Substituting these values we find for the speed at P,

$$\begin{aligned} v^2 &= \mu(a^2 + b^2 - r^2) \\ &= \mu \,.\, OD^2, \end{aligned}$$

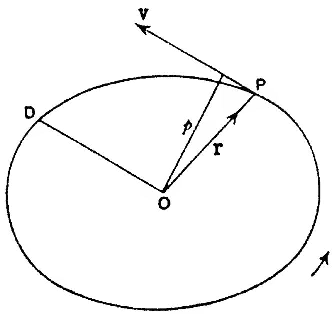

FIG. 54.

where OD is the semi-diameter conjugate to OP, and therefore parallel to the tangent at P. Thus

$$\mathbf{v} = \sqrt{\mu} \,.\, \overrightarrow{OD}$$

is the velocity at the point P.

The *periodic time* is the time to describe the whole area of the ellipse at the rate $\frac{h}{2}$. This is

$$T = \frac{2\pi ab}{h} = \frac{2\pi ab}{\sqrt{\mu}\,.\,ab} = \frac{2\pi}{\sqrt{\mu}}.$$

The period T thus depends only on the intensity μ of the force, and not on the size of the orbit.

If $\mathbf{r}$ is expressed in terms of two constant unit vectors $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$ in the plane of the orbit, as

$$\mathbf{r} = x\hat{\mathbf{a}} + y\hat{\mathbf{b}},$$

the equation of motion (1) becomes

$$\ddot{x}\hat{\mathbf{a}} + \ddot{y}\hat{\mathbf{b}} = -\mu(x\hat{\mathbf{a}} + y\hat{\mathbf{b}}),$$

and is equivalent to the two scalar equations

$$\ddot{x} = -\mu x; \quad \ddot{y} = -\mu y,$$

which represent simple harmonic variations of x, y of common period $2\pi/\sqrt{\mu}$. The velocity of P in the ellipse is

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{x}\hat{\mathbf{a}} + \dot{y}\hat{\mathbf{b}}.$$

Thus two simple harmonic motions with a common period and centre, but in different directions and with different phases, compound into elliptic motion with the same period and the same centre. (Cf. also Exercise 3 at end of chapter.)

81.* Motion of a particle on a fixed curve. Consider next a particle moving along a fixed curve, *e.g.* a bead sliding along a fixed wire. Let v denote its speed at any point and $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$ the unit vectors in the directions of the tangent, principal normal and binormal at that point. Given the external force $\mathbf{F}$ acting on the particle, it may be resolved into components in these directions,

$$\mathbf{F} = F_1\mathbf{t} + F_2\mathbf{n} + F_3\mathbf{b},$$

while the action of the curve on the particle may be specified by the sum of two forces; one

$$\mathbf{R} = R_2\mathbf{n} + R_3\mathbf{b}$$

perpendicular to the curve, and the other a frictional force parallel to the tangent, and equal in magnitude to μ times the former, where μ is the coefficient of friction. The frictional force

is thus $-\mu R\mathbf{t}$, having the opposite direction to the velocity $\mathbf{v}=v\mathbf{t}$. The acceleration of the particle, as shown in Art. 67, is

$$\mathbf{a}=\frac{dv}{dt}\mathbf{t}+\kappa v^2\mathbf{n},$$

having zero resolute in the direction of $\mathbf{b}$. The equation of motion for the particle is therefore

$$m\mathbf{a}=\mathbf{F}+\mathbf{R}-\mu R\mathbf{t},$$

that is $$m(\dot{v}\mathbf{t}+\kappa v^2\mathbf{n})=(F_1\mathbf{t}+F_2\mathbf{n}+F_3\mathbf{b})+(R_2\mathbf{n}+R_3\mathbf{b})$$
$$-\mu\sqrt{R_2^2+R_3^2}\,\mathbf{t}.$$

This is equivalent to the three scalar equations

$$m\dot{v}=F_1-\mu\sqrt{R_2^2+R_3^2}. \quad\text{.........................(1)}$$
$$m\kappa v^2=F_2+R_2. \quad\text{...(2)}$$
$$0=F_3+R_3. \quad\text{...(3)}$$

Thus the resolutes of the external force and the reaction of the curve in the direction of $\mathbf{b}$ are equal and opposite.

For a *smooth curve* $\mu=0$ and the equation (1) is simply $m\dfrac{dv}{dt}=F_1$. If F_1 is given as a function of s, viz. $F_1(s)$, we may write

$$\frac{dv}{dt}=\frac{dv}{ds}\frac{ds}{dt}=v\frac{dv}{ds}=\frac{d}{ds}(\tfrac{1}{2}v^2),$$

and the equation becomes

$$m\frac{d}{ds}(\tfrac{1}{2}v^2)=F_1(s),$$

which, on integration, gives the formula

$$\tfrac{1}{2}m(v^2-v_0^2)=\int_A^s F_1(s)ds,$$

v_0 being the speed at A. This result is simply that the increase in the kinetic energy of the particle is equal to the work done by the external force on the particle. Having determined v we find R_2 from the equation (2), while R_3 is known from (3).

When the external force is at each point parallel to the plane of curvature, $F_3=0$, and therefore $R_3=0$ whether the curve is rough or smooth. The equations of motion are then

$$\begin{cases}\tfrac{1}{2}m\dfrac{dv^2}{ds}=F_1-\mu R_2,\\ m\kappa v^2=F_2+R_2. \quad\text{.........................(2)}\end{cases}$$

Elimination of R_2 gives

$$\tfrac{1}{2}m\frac{dv^2}{ds} + m\mu\kappa v^2 = F_1 + \mu F_2,$$

which is a linear differential equation of the first order in v^2, whose solution is found in the usual way. Then, having determined v^2 we find R_2 from (2).

Note. Further worked examples will be found among the following exercises.

EXERCISES ON CHAPTER VI.

1. Three particles A, B, C at the points $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$ are moving with velocities $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ respectively. Find the rate of change of the vector area of the triangle ABC.

2. Four particles A, B, C, D at the points $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$, $\mathbf{r}_4$ are moving with velocities $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$ respectively. Find the rate of change of the volume of the tetrahedron $ABCD$.

3. If $\mathbf{a}$, $\mathbf{b}$ are constant vectors and t the time variable, show that a particle whose position vector at any instant is

$$\mathbf{r} = \cos nt\,\mathbf{a} + \sin nt\,\mathbf{b}$$

is moving in an ellipse whose centre is the origin; and that the motion is that due to a central force varying as the distance.

4. Show that, when a particle moves on a smooth curve under the action of gravity, its speeds u, v at any two points P, Q are connected by

$$v^2 = u^2 + 2gh,$$

where h is the vertical depth of Q below P.

5. **Hodograph.** The hodograph of a moving point P is the locus of another point Q whose position vector at any instant is equal (or proportional) to the velocity vector of P.

Suppose, for instance, that P is moving with *constant acceleration* $g\mathbf{d}$ (Art. 70). Then its velocity at any instant is

$$\mathbf{v} = \mathbf{v}_0 + tg\mathbf{d}.$$

Hence the position vector of Q is equal (or proportional) to

$$\mathbf{R} = \mathbf{v}_0 + tg\mathbf{d}.$$

Thus the locus of Q is a straight line parallel to $\mathbf{d}$. And since t increases uniformly, Q moves uniformly along this straight line. The velocity of Q is equal (or proportional) to $\dot{\mathbf{R}} = g\mathbf{d}$; that is to the acceleration of P. This is always true. For, since $\mathbf{R} = c\mathbf{v}$, it follows that $\dot{\mathbf{R}} = c\dot{\mathbf{v}} = c\mathbf{a}$, where $\mathbf{a}$ is the acceleration of P.

If the motion of P is that due to a *central force varying as the distance*, then (Art. 80) P moves in an ellipse whose centre is at O, and its velocity at any instant is $\mathbf{v} = \sqrt{\mu} \,.\, \overrightarrow{OD}$, where OD is the semi-diameter conjugate to OP. Thus, for the point Q on the hodograph, $\mathbf{R}$ is proportional to $\overrightarrow{OD}$, showing that the hodograph is a similar ellipse described in the same periodic time. The acceleration of P is proportional to the velocity of Q, and therefore to $\overrightarrow{PO}$.

6. A particle describes a circle with uniform speed. Show that the hodograph is a circle described uniformly.

7. A bead slides down the circumference of a smooth vertical circle, starting from rest at the highest point. Show that the equation of the hodograph in polar coordinates is $r = c \sin \frac{\theta}{2}$.

8. *If a particle describes a conic under a central force to the focus, the hodograph is a circle.*

From equation (2) of Art. 78, on transposing and squaring we find

$$\hat{\mathbf{r}}^2 = \frac{h^2}{\mu^2}\mathbf{v}^2 - 2\frac{he}{\mu}\hat{\mathbf{a}}\cdot\mathbf{v}\times\mathbf{k} + e^2,$$

that is
$$\mathbf{v}^2 - 2\mathbf{v}\cdot\left(\frac{e\mu}{h}\mathbf{k}\times\hat{\mathbf{a}}\right) + (e^2 - 1)\frac{\mu^2}{h^2} = 0.$$

Hence the hodograph is the circle

$$\mathbf{R}^2 - 2\mathbf{R}\cdot\mathbf{c} + (e^2 - 1)\frac{\mu^2}{h^2} = 0,$$

whose centre is the point $\mathbf{c} = \frac{e\mu}{h}\mathbf{k}\times\hat{\mathbf{a}}$. If the conic is a parabola, $e = 1$, and the hodograph passes through the origin.

9. A particle describes an ellipse under a force to the centre. Show that its angular velocity about a focus varies inversely as its distance from that focus; and that the sum of the reciprocals of its angular velocities about the two foci is constant.

10. Show that the path of a point P, whose velocity is the sum of two components of constant magnitude u, v, the first in a fixed direction and the second perpendicular to the line joining P to a fixed point S, is a conic with focus at S and eccentricity $\frac{u}{v}$.

11. A particle describes an ellipse with a focus as the centre of force. Show that the speed at the end of the minor axis is a mean proportional between the speeds at the ends of any diameter.

12. In the previous exercise, show that the angular velocity of the particle about the other focus varies inversely as the square of the normal.

13. A particle is describing an ellipse under a central force to the focus. When it reaches the end of the minor axis the intensity of force is diminished by one-third. Find the position and size of the new orbit, and show that the join of its centre and focus is bisected by the minor axis of the original orbit.

14. If, in the previous exercise, the change at the end of the minor axis is an alteration of the law of force to variation directly as the distance, the magnitude of the force at that point remaining the same, show that the periodic time is unaltered, and that the sum of the new axes is to their difference as the sum of the old axes to the distance between the foci.

15. A particle describes an ellipse with one focus S as the centre of force. When the particle is at P the centre of force is suddenly removed to the other focus S'. If κ, κ' are the curvatures at P of the old and new orbits, show that

$$\kappa : \kappa' = S'P^2 : SP^2.$$

16. Show that the rate of rotation of the direction of motion of a particle in an ellipse under a force to the focus, is a maximum or minimum when the particle is furthest from that focus, according as the eccentricity is greater or less than $\frac{1}{3}$.

17. In elliptic motion under a force to the focus, when the particle arrives at P the direction of motion is turned through a right angle, the speed being unaltered. Show that the particle will then describe an ellipse whose eccentricity varies as the distance of P from the centre.

18. Central force any function of the distance. Suppose a particle moving under a central force $-F\hat{\mathbf{r}}$ per unit mass, where F is a function of the distance r from the centre of force. Then the equation of motion is

$$\frac{d^2\mathbf{r}}{dt^2} = -F\hat{\mathbf{r}},$$

and forming the scalar product of each side with $2\frac{d\mathbf{r}}{dt}$ we find

$$2\frac{d\mathbf{r}}{dt}\cdot\frac{d^2\mathbf{r}}{dt^2} = -2F\hat{\mathbf{r}}\cdot\frac{d\mathbf{r}}{dt} = -2F\frac{dr}{dt}.$$

On integration this becomes

$$\left(\frac{d\mathbf{r}}{dt}\right)^2 = C - 2\int F\frac{dr}{dt}dt$$

or

$$v^2 = C - 2\int F\,dr.$$

This equation gives the value of the speed at any distance. Writing h^2/p^2 instead of v^2, and differentiating the last equation with respect to r, we find

$$\frac{h^2}{p^3}\frac{dp}{dr}=F.$$

This is the p, r differential equation of the orbit when the function F is given. Or, given the orbit, this equation tells the law of force to the pole (origin) under which the orbit can be described.

For instance, the p, r equation of an ellipse referred to a focus is

$$\frac{1}{p^2}=\frac{a}{b^2}\left(\frac{2}{r}-\frac{1}{a}\right).$$

Hence, for a particle describing an elliptic orbit under a central force to the focus,

$$F=\frac{h^2}{p^3}\frac{dp}{dr}=\frac{h^2a}{b^2}\cdot\frac{1}{r^2}.$$

The law of force is therefore that of the inverse square.

19. Find the law of force to the centre of an ellipse under which the ellipse will be described.

This is the reverse of Art. 80. For an ellipse, with origin at the centre,

$$\frac{1}{p^2}=\frac{a^2+b^2-r^2}{a^2b^2},$$

whence

$$F=\frac{h^2}{p^3}\frac{dp}{dr}=\frac{h^2}{a^2b^2}r,$$

varying directly as the distance.

20. Show that the law of force to the pole under which the equiangular spiral $p=r\sin\alpha$ can be described is that of the inverse cube. Also that the speed varies inversely as r.

21. Prove that the hodograph of the motion in the previous exercise is also an equiangular spiral.

22. Show that the force under which a particle describes a circle, with the centre of force on the circumference, is $F=\mu/r^5$, and that the speed varies inversely as r^2.

23. Show that the hodograph of the motion in the previous exercise is a parabola.

24. A particle is describing a parabolic orbit (latus rectum $4a$) about a centre of force (μ) in the focus; and when it arrives at a distance r from the focus, moving toward the vertex, the centre of force ceases to act for a certain interval T. Prove that when the force operates again the new orbit will be an ellipse, parabola or hyperbola according as $T <, =, > 2r\sqrt{(r-a)/2\mu}$.

25. A smooth wire in the form of a circular helix has its axis vertical, and a bead slides down it under gravity. Determine the speed at a depth z, and the action on the wire; and show that the time to fall this depth is cosec $\alpha\sqrt{2z/g}$.

26. Show that the hodograph of the motion in the previous exercise is a curve described on the surface of a right circular cone of semi-vertical angle $\frac{\pi}{2}-\alpha$.

27. A particle P moves on a smooth helix (a, α) under the action only of a central force to a fixed point O on the axis equal to $\mu m \,.\, OP$, where m is the mass of the particle. Show that the action on the curve cannot vanish unless the maximum speed of the particle is $a\sqrt{\mu}\,.\,\sec\alpha$.

28. If the hodograph be a circle described with constant angular velocity about a point on its circumference, show that the path of the particle is a cycloid.

29. P is a point on the tangent at a variable point Q to a fixed circle of radius a; QP is of length r, and makes an angle θ with a fixed tangent. Show that the resolutes of the acceleration of P along and perpendicular to QP are

$$\ddot{r}-r\dot{\theta}^2+a\ddot{\theta} \quad \text{and} \quad \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})+a\dot{\theta}^2.$$

30. A particle of unit mass is placed in a smooth tube in the form of an equiangular spiral of angle α, and starts from rest at a distance $2d$ under a force μ/r^3 to the pole. Show that it will reach the pole in a time $\pi\sec\alpha\sqrt{d^4/\mu}$.

31. A particle, constrained to move on a circular wire, is acted on by a central force to a point on the circumference varying inversely as the fifth power of the distance. Prove that the action on the wire is constant in magnitude.

CHAPTER VII.

DYNAMICS OF A SYSTEM OF PARTICLES AND OF A RIGID BODY.

82. Linear Momentum. Consider a system of moving particles, either independent of each other or under any sort of mutual action. Their relative positions may be changing, or they may be rigidly attached to one another as in the case of a rigid body. Let m be the mass of any one of the particles, $\mathbf{r}$ its position vector relative to a fixed origin O, and $\mathbf{v} = \dot{\mathbf{r}}$ its velocity. Then the linear momentum of the particle is $m\mathbf{v}$. We define the linear momentum of the system as the vector sum of the linear momenta of the separate particles. It is therefore represented by the vector

$$\mathbf{M} = \Sigma m\dot{\mathbf{r}} = \frac{d}{dt}\Sigma m\mathbf{r}.$$

But the centre of mass of the system has a position vector $\bar{\mathbf{r}}$ given by

$$M\bar{\mathbf{r}} = \Sigma m\mathbf{r},$$

where $M = \Sigma m$ is the mass of the whole system. Hence

$$\mathbf{M} = \frac{d}{dt}(M\bar{\mathbf{r}}) = M\frac{d\bar{\mathbf{r}}}{dt} = M\bar{\mathbf{v}}, \quad \text{.........................(1)}$$

$\bar{\mathbf{v}}$ denoting the velocity of the centre of mass. Thus *the linear momentum of the system is equal to that of a single particle of mass equal to the total mass of the system, moving with the velocity of the c.m.*

The rate of increase of the linear momentum of the system is given by

$$\frac{d\mathbf{M}}{dt} = \frac{d}{dt}(M\bar{\mathbf{v}}) = M\bar{\mathbf{a}}, \quad \text{..............................(2)}$$

where $\bar{\mathbf{a}}$ is the acceleration of the c.m.

83. Equation of motion of the c.m. Let $\mathbf{F}$ be the resultant force acting on the particle m. Then the equation of motion for the particle is

$$\mathbf{F} = \frac{d}{dt}(m\mathbf{v}).$$

Taking the vector sum of the resultant forces on all the particles, we have

$$\Sigma\mathbf{F} = \frac{d}{dt}\Sigma m\mathbf{v} = \frac{d\mathbf{M}}{dt}. \quad \text{.............................(3)}$$

Now the vector sum $\Sigma\mathbf{F}$ includes both forces whose origin is external to the system, and forces of internal action between the separate particles of the system. But if we assume, in accordance with Newton's Third Law of Motion, that the internal action between the particles is represented by pairs of equal and opposite forces, the vector sum of these is zero. In calculating $\Sigma\mathbf{F}$ we need therefore only consider the external forces acting on the particles of the system. The equation (3) then shows that *the vector sum of the external forces acting on the system is equal to the rate of increase of its linear momentum.*

The equation may also be written

$$\Sigma\mathbf{F} = \frac{d}{dt}(M\bar{\mathbf{v}}) = M\bar{\mathbf{a}}, \quad \text{.............................(4)}$$

so that the c.m. has the same acceleration as a particle of mass equal to the total mass of the system, acted on by a force equal to the vector sum of all the external forces.

84. Angular Momentum. The moment of momentum, or angular momentum, of the particle m about the origin O (cf. Art. 76) is represented by the vector $m\mathbf{r}\times\mathbf{v}$, and its rate of change by $m\mathbf{r}\times\dot{\mathbf{v}}$. The angular momentum of the system about O is defined as the vector sum of the angular momenta of the separate particles about O. Representing it by $\mathbf{H}$, we have

$$\mathbf{H} = \Sigma\mathbf{r}\times m\mathbf{v}, \quad \text{.....................................(5)}$$

and its rate of change is

$$\frac{d\mathbf{H}}{dt} = \Sigma\mathbf{r}\times\frac{d}{dt}(m\mathbf{v}) = \Sigma\mathbf{r}\times\mathbf{F}, \quad \text{...........................(6)}$$

where $\mathbf{F}$ is the resultant force acting on the particle at $\mathbf{r}$. The product $\mathbf{r}\times\mathbf{F}$ represents the (vector) moment or torque of the force $\mathbf{F}$ about O. In the summation the internal actions may

be neglected, since each pair of equal and opposite collinear forces has zero moment about O. The equation (6) then states that *the rate of increase of the A.M. of the system about a fixed point O is equal to the vector sum of the torques about O of all the external forces acting on the system.*

The angular momentum of the system about a fixed line through O in the direction of the unit vector $\hat{\mathbf{c}}$, is the (scalar) resolute in this direction of the A.M. $\mathbf{H}$. It is therefore given by $\hat{\mathbf{c}}\cdot\mathbf{H}$, and its rate of change by $\frac{d}{dt}(\hat{\mathbf{c}}\cdot\mathbf{H}) = \hat{\mathbf{c}}\cdot\frac{d\mathbf{H}}{dt}$. Further, by Art. 40, the sum of the moments about this line of all the external forces acting on the system is

$$\Sigma\hat{\mathbf{c}}\cdot\mathbf{r}\times\mathbf{F} = \hat{\mathbf{c}}\cdot\Sigma\mathbf{r}\times\mathbf{F} = \hat{\mathbf{c}}\cdot\frac{d\mathbf{H}}{dt}.$$

Hence the rate of increase of the A.M. of the system about a fixed line is equal to the sum of the moments of the external forces about that line.

In calculating the A.M. of the system about any point O, it is frequently convenient to use the velocity of the centre of mass G, and the velocity of the particles relative to G. If $\bar{\mathbf{r}}$, $\bar{\mathbf{v}}$ are the position vector and velocity of the c.m., and $\mathbf{r}'$, $\mathbf{v}'$ those of the particle m relative to G,

$$\mathbf{r} = \bar{\mathbf{r}} + \mathbf{r}' \quad \text{and} \quad \mathbf{v} = \bar{\mathbf{v}} + \mathbf{v}'.$$

Hence the A.M. about O is

$$\begin{aligned}\mathbf{H} &= \Sigma m\mathbf{r}\times\mathbf{v} = \Sigma m(\bar{\mathbf{r}} + \mathbf{r}')\times(\bar{\mathbf{v}} + \mathbf{v}')\\ &= \Sigma m\bar{\mathbf{r}}\times\bar{\mathbf{v}} + \Sigma m\mathbf{r}'\times\mathbf{v}' + (\Sigma m\mathbf{r}')\times\bar{\mathbf{v}} + \bar{\mathbf{r}}\times\Sigma m\mathbf{v}'.\end{aligned}$$

Now the last two terms vanish; for $\Sigma m\mathbf{r}'$ is constantly zero, and therefore also its derivative $\Sigma m\mathbf{v}'$. The equation may then be written

$$\mathbf{H} = \bar{\mathbf{r}}\times M\bar{\mathbf{v}} + \Sigma\mathbf{r}'\times m\mathbf{v}'.$$

Thus the A.M. of the system about O is the vector sum of the A.M. about O of a particle of mass M at G moving with the velocity of G, and the A.M. about G of the system in its motion relative to G.

Example.

The motion of a body is given by the velocity $\bar{\mathbf{v}}$ of its c.m. and the A.M. $\mathbf{H}$ about that point. Find the A.M. about a straight line through the point A parallel to the unit vector $\mathbf{b}$.

Let M be the mass of the body and $\mathbf{a}$ the position vector of A relative to the c.m. Then $-\mathbf{a}$ is that of the c.m. relative to A; and by the theorem just proved the A.M. about the point A is

$$\mathbf{H} + (-\mathbf{a}) \times M\bar{\mathbf{v}}.$$

Therefore the (scalar) A.M. about an axis through A parallel to $\mathbf{b}$ is

$$\mathbf{b} \cdot (\mathbf{H} - M\mathbf{a} \times \bar{\mathbf{v}}) = \mathbf{b} \cdot \mathbf{H} + M[\mathbf{a}\mathbf{b}\bar{\mathbf{v}}].$$

85. Moving origin of moments. Instead of the fixed point O, suppose we choose as origin of moments a moving point O', whose variable position vector relative to O is $\mathbf{s}$. Then if $\mathbf{r}$, $\mathbf{r}'$ are the position vectors of the particle m relative to O, O' respectively, $\mathbf{r} = \mathbf{s} + \mathbf{r}'$. Let $\mathbf{H}$, $\mathbf{H}'$ represent the angular momenta of the system about O, O' respectively. Then

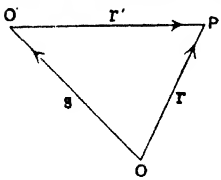

FIG. 55.

$$\begin{aligned}\mathbf{H} &= \Sigma\mathbf{r} \times m\mathbf{v} = \Sigma(\mathbf{s} + \mathbf{r}') \times m\mathbf{v} \\ &= \mathbf{s} \times \Sigma m\mathbf{v} + \Sigma\mathbf{r}' \times m\mathbf{v} \\ &= \mathbf{s} \times \mathbf{M} + \mathbf{H}',\end{aligned}$$

where $\mathbf{M}$ is the linear momentum of the system. By differentiation we find for the rate of change of $\mathbf{H}$,

$$\begin{aligned}\frac{d\mathbf{H}}{dt} &= \frac{d\mathbf{H}'}{dt} + \frac{d\mathbf{s}}{dt} \times \mathbf{M} + \mathbf{s} \times \frac{d\mathbf{M}}{dt} \\ &= \frac{d\mathbf{H}'}{dt} + \mathbf{v}' \times \mathbf{M} + \mathbf{s} \times \Sigma\mathbf{F},\end{aligned}$$

in which $\mathbf{v}' \equiv \dot{\mathbf{s}}$ is the velocity of O' relative to O.

But the rate of increase of the A.M. about O is equal to the torque of the external forces about O. If $\mathbf{L}$ represents this torque, and $\mathbf{L}'$ the torque about O', we have

$$\begin{aligned}\mathbf{L} &= \Sigma\mathbf{r} \times \mathbf{F} = \Sigma(\mathbf{s} + \mathbf{r}') \times \mathbf{F} \\ &= \mathbf{s} \times \Sigma\mathbf{F} + \mathbf{L}'.\end{aligned}$$

Equating this value of $\mathbf{L}$ to the value found above for $\frac{d\mathbf{H}}{dt}$, we find

$$\frac{d\mathbf{H}'}{dt} = \mathbf{L}' - \mathbf{v}' \times \mathbf{M}, \qquad \text{.................................(7)}$$

which is the relation between the rate of change of the A.M. about a moving point, and the torque of the external forces about that point.

A case of particular interest is that for which the moving point O' is the c.m. of the system. For then $\mathbf{v}' = \bar{\mathbf{v}}$ and $\mathbf{M} = M\bar{\mathbf{v}}$, so that the last term in (7) vanishes. Hence with the c.m. as origin of moments, whether moving or at rest, the relation $\frac{d\bar{\mathbf{H}}}{dt} = \mathbf{L}$ is always true.

In calculating $\bar{\mathbf{H}}$ we may treat the c.m. as a fixed point, and consider only velocities relative to it. For if $\bar{\mathbf{v}}$ is the velocity of the c.m., and $\mathbf{v}'$ that of the particle m relative to it,

$$\begin{aligned}\bar{\mathbf{H}} &= \Sigma\mathbf{r}'\times m(\bar{\mathbf{v}} + \mathbf{v}')\\ &= (\Sigma m\mathbf{r}')\times\bar{\mathbf{v}} + \Sigma\mathbf{r}'\times m\mathbf{v}'.\end{aligned}$$

But the first term is zero, since $\Sigma m\mathbf{r}' = 0$. Therefore

$$\bar{\mathbf{H}} = \Sigma\mathbf{r}'\times m\mathbf{v}',$$

and depends only on the motion of the system relative to the c.m.

86. Equations for impulsive forces. Suppose the system acted on by a set of impulsive forces. When there are any connections between the particles of the system, these forces will in general cause impulsive action between them. But we assume, in accordance with Newton's Third Law, that such action between two particles consists of a pair of equal and opposite impulsive forces along the line joining them. Let $\mathbf{v}$ and $\mathbf{v}_0$ be the velocities of the particle m just after and just before the blow. Then the resultant impulsive force $\mathbf{I}$ on this particle is equal to the change of momentum it produces in the particle (Art. 73): or

$$\mathbf{I} = m(\mathbf{v} - \mathbf{v}_0).$$

Taking the vector sum of the impulsive forces on all the particles, we have

$$\Sigma\mathbf{I} = \Sigma m(\mathbf{v} - \mathbf{v}_0) = \mathbf{M} - \mathbf{M}_0, \quad\text{......(8)}$$

where $\mathbf{M}$, $\mathbf{M}_0$ are the linear momenta of the system just after and just before the impulsive forces act. In forming the sum $\Sigma\mathbf{I}$ we may neglect the internal impulses which consist of equal and opposite pairs. The equation (8) then shows that *the vector sum of the external impulsive forces acting on the system is equal to the increase of linear momentum produced in the system.*

Similarly, taking moments about a fixed point O, relative to which the position vector of the particle m is $\mathbf{r}$, we have

$$\Sigma\mathbf{r}\times\mathbf{I} = \Sigma\mathbf{r}\times m(\mathbf{v} - \mathbf{v}_0) = \mathbf{H} - \mathbf{H}_0, \quad\text{......(9)}$$

where $\mathbf{H}_0$ and $\mathbf{H}$ are the A.M. of the system about O before and

after the blow. In the summation on the left-hand side we may neglect the internal impulses for the same reason as before, thus obtaining the result that *the vector sum of the moments of the external impulsive forces about a fixed point is equal to the increase in A.M. produced about that point.*

Kinematics of a Rigid Body.

87. Motion about a fixed point. *Any displacement of a rigid body with one point O fixed is equivalent to a rotation about a definite axis through O.*

To prove this, consider a unit sphere fixed in the body and having its centre at O. As the body moves, this sphere turns

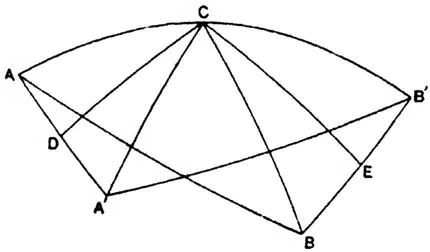

FIG. 56.

about its centre; and the motion of any point P of the body is determined by the motion of that point of the spherical surface which lies in the line OP. We may thus think of the body as a sphere of unit radius, and consider the motion of points on the surface of this sphere. Suppose that during any displacement the point A moves to A' and the point B to B'. Join A, A' and also B, B' by arcs of great circles, and bisect these arcs at right angles by other arcs of great circles, DC and EC, intersecting at C. Then clearly the arcs AC, $A'C$ are equal, and also the arcs BC, $B'C$. But since the particles of a rigid body remain at the same distance apart, the arcs AB and $A'B'$ are equal. Hence the spherical triangles ACB and $A'CB'$ are congruent. Thus the portion of the spherical surface which originally occupied the position ACB, occupies the position $A'CB'$ after the displacement. The point C of the body occupies its

original position, and the same is therefore true of all points on the line OC. The displacement is therefore equivalent to a rotation about the axis OC through an angle

$$ACA' = BCB'.$$

Suppose now that the displacement is one which takes place in a short interval δt owing to a continuous motion of the body, and that $\delta\theta$ is the circular measure of the small angle ACA' of rotation about OC. The average angular velocity during the interval is $\frac{\delta\theta}{\delta t}$ radians per unit time about OC. If now δt tends to zero, the limiting value ω of this average angular velocity is called the *instantaneous angular velocity*, and the limiting position of OC the *instantaneous axis* of rotation. In the case of a body turning about a fixed point O, the instantaneous angular velocity may be completely specified by a vector $\mathbf{A}$, whose module is ω, and whose direction is parallel to the instantaneous axis and in the positive sense relative to the rotation.

The instantaneous velocity of any particle of the body may now be found as in Art. 41. If $\mathbf{r}$ is the position vector of the particle relative to O, its instantaneous velocity is perpendicular to the plane of $\mathbf{r}$ and $\mathbf{A}$. The particle is moving for the moment in a circle whose centre is the projection N of the particle on the instantaneous axis (Fig. 39); and its speed is therefore

$$\omega \,.\, PN = \omega r \sin P\hat{O}N.$$

Hence the instantaneous velocity of the particle is

$$\mathbf{v} = \mathbf{A}{\times}\mathbf{r}.$$

Example.

A rigid body is turning about a fixed point O with angular velocity $\mathbf{A}$. If $\mathbf{F}$ is the vector sum of the external forces, and $\mathbf{r}$ the position vector of the c.m. relative to O, show that the force $\mathbf{R}$ of constraint at O is given by

$$\mathbf{F} + \mathbf{R} = M\frac{d\mathbf{A}}{dt}{\times}\bar{\mathbf{r}} + M\mathbf{A}{\times}(\mathbf{A}{\times}\mathbf{r}).$$

The velocity of the c.m. is $\mathbf{A}{\times}\mathbf{r}$, and its acceleration therefore

$$\bar{\mathbf{a}} = \frac{d}{dt}(\mathbf{A}{\times}\bar{\mathbf{r}}) = \frac{d\mathbf{A}}{dt}{\times}\bar{\mathbf{r}} + \mathbf{A}{\times}\frac{d\bar{\mathbf{r}}}{dt}$$

$$= \frac{d\mathbf{A}}{dt}{\times}\bar{\mathbf{r}} + \mathbf{A}{\times}(\mathbf{A}{\times}\bar{\mathbf{r}}),$$

since $\frac{d\bar{\mathbf{r}}}{dt}$ is the velocity of the c.m. Equating the vector sum of all the forces on the body to $M\bar{\mathbf{a}}$ (cf. Art. 83) we have the required result.

88. General motion of a rigid body. Suppose now that no point of the body is fixed. Then the most general displacement of the body is equivalent to a translation in which all particles have the same displacement, together with a rotation about some definite axis. For the body may be translated without rotation so as to bring any one point O into its final position; and then the whole body may be brought into its final position by rotation about some axis through O.

Consider the small displacement that takes place in a short interval δt owing to the finite velocity of the body. This is equivalent to a translation $\delta\mathbf{s}$ of every particle, such that a certain point O is brought into its final position, together with a rotation $\delta\theta$ about an axis through O parallel to some unit vector $\hat{\mathbf{a}}$. Then $\frac{\delta\mathbf{s}}{\delta t}$ represents the average velocity of O during the interval, and $\frac{\delta\theta}{\delta t}\hat{\mathbf{a}}$ the average angular velocity of the body about O. Taking limiting values as $\delta t \to 0$, we have $\mathbf{v} = \frac{d\mathbf{s}}{dt}$ for the instantaneous velocity of O, and $\mathbf{A} = \mathrm{Lt}\left(\frac{\delta\theta}{\delta t}\hat{\mathbf{a}}\right)$ for the instantaneous angular velocity about O. During the interval δt the displacement of the point P, whose position vector is $\mathbf{r}$ relative to O, is $\delta\mathbf{s} + \delta\theta\hat{\mathbf{a}}\times\mathbf{r}$, the first term being due to the translation, and the second to the rotation $\delta\theta$. The average velocity of this point during the interval is therefore $\frac{\delta\mathbf{s}}{\delta t} + \frac{\delta\theta}{\delta t}\hat{\mathbf{a}}\times\mathbf{r}$. Taking limiting values we have for the instantaneous velocity of P

$$\mathbf{V} = \mathbf{v} + \mathbf{A}\times\mathbf{r}. \qquad \text{.................................(10)}$$

The first term is the velocity of O, the second is the velocity of P relative to O.

Consider another point O' whose position vector relative to O is $\mathbf{s}$, and whose velocity $\mathbf{v}'$ is therefore given by

$$\mathbf{v}' = \mathbf{v} + \mathbf{A}\times\mathbf{s}.$$

The position vector of P relative to O' is $\mathbf{r}' = \mathbf{r} - \mathbf{s}$ (Fig. 55), and the velocity of P given by (10) may be expressed,

$$\begin{aligned}\mathbf{V} &= (\mathbf{v} + \mathbf{A}\times\mathbf{s}) + \mathbf{A}\times(\mathbf{r} - \mathbf{s}) \\ &= \mathbf{v}' + \mathbf{A}\times\mathbf{r}'.\end{aligned}$$

This is of the same form as (10), showing that the velocity of P relative to O' is that due to an angular velocity $\mathbf{A}$ of the body about O'. *The value* $\mathbf{A}$ *of the angular velocity is thus the same for all origins, and is a property of the body as a whole.* The formula (10) will be found very useful for writing down the velocity of any point of the body.

If the point O' is such that its velocity is parallel to the axis of the angular velocity, $\mathbf{v}'$ is parallel to $\mathbf{A}$, and therefore $\mathbf{A}\times\mathbf{v}' = 0$ That is

$$\mathbf{A}\times(\mathbf{v} + \mathbf{A}\times\mathbf{s}) = 0,$$

$$\mathbf{A}\times\mathbf{v} + \mathbf{A}\cdot\mathbf{s}\mathbf{A} - A^2\mathbf{s} = 0,$$

from which it follows that

$$\mathbf{s} = \frac{\mathbf{A}\times\mathbf{v}}{A^2} + u\mathbf{A}, \quad \text{.........................(11)}$$

where, by substitution, it is found that the value of u is arbitrary. Thus the locus of O' is a straight line parallel to $\mathbf{A}$ passing through the point $\mathbf{A}\times\mathbf{v}/A^2$. The velocity of any point on this line is parallel to $\mathbf{A}$. The instantaneous motion of the body is therefore equivalent to a *screw motion* about this line, which is called the *axis of the screw*. Every point on the axis is moving along the axis, while the body is turning round it with an angular velocity $\mathbf{A}$.

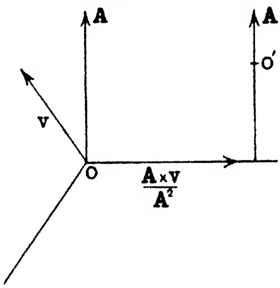

FIG. 57

From (10) it follows that $\mathbf{V}\cdot\mathbf{A} = \mathbf{v}\cdot\mathbf{A}$, and this expression has therefore the same value for all points of the body. It is called an *invariant* of the motion and will be denoted by Γ. This invariant property is simply that the resolute of the velocity in the direction of $\mathbf{A}$ is the same for all points. We have already seen that A^2 is itself an invariant, being the same for all origins.

The *pitch* of the screw motion is the advance along the axis per radian of rotation of the body. Since, for a point O' on the axis of the screw, $\mathbf{v}'$ has the same direction as $\mathbf{A}$, the pitch p is given by

$$p = \mathbf{v}' : \mathbf{A} = \mathbf{v}' \cdot \mathbf{A} : \mathbf{A}^2 = \frac{\Gamma}{\mathbf{A}^2}.$$

The pitch, as thus defined, is positive if $\mathbf{A}$ and $\mathbf{v}'$ have the same sense; that is, if the screw is right-handed. It is negative if the screw is left-handed.

89. Simultaneous motions. A body is said to possess several motions simultaneously, when the velocity of each particle is the vector sum of the velocities it would have due to each motion separately. If, for instance, the body possesses simultaneous angular velocities $\mathbf{A}_1$ and $\mathbf{A}_2$ about a fixed point O, the particle whose position vector relative to O is $\mathbf{r}$ has a velocity

$$\mathbf{A}_1 \times \mathbf{r} + \mathbf{A}_2 \times \mathbf{r} = (\mathbf{A}_1 + \mathbf{A}_2) \times \mathbf{r}.$$

But this is the velocity of the particle due to a single angular velocity $\mathbf{A}_1 + \mathbf{A}_2$ about O. Hence *simultaneous angular velocities about a fixed point are compounded by the law of vector addition.* The argument is clearly true for any number of angular velocities.

Suppose that the body has simultaneous angular velocities

$$\mathbf{A}_1 = \omega_1 \hat{\mathbf{a}} \quad \text{and} \quad \mathbf{A}_2 = \omega_2 \hat{\mathbf{a}}$$

about parallel axes through the points $\mathbf{s}_1$ and $\mathbf{s}_2$ respectively. Then the velocity of the particle at $\mathbf{r}$ is the vector sum of the velocities due to each; that is

$$\mathbf{A}_1 \times (\mathbf{r} - \mathbf{s}_1) + \mathbf{A}_2 \times (\mathbf{r} - \mathbf{s}_2) = (\mathbf{A}_1 + \mathbf{A}_2) \times \left(\mathbf{r} - \frac{\omega_1 \mathbf{s}_1 + \omega_2 \mathbf{s}_2}{\omega_1 + \omega_2}\right),$$

showing that the motion is equivalent to an angular velocity $\mathbf{A}_1 + \mathbf{A}_2$ about a parallel axis through the point

$$(\omega_1 \mathbf{s}_1 + \omega_2 \mathbf{s}_2)/(\omega_1 + \omega_2),$$

which divides the line joining $\mathbf{s}_1$ and $\mathbf{s}_2$ in the ratio $\omega_2 : \omega_1$. If, however, the angular velocities $\mathbf{A}_1$ and $\mathbf{A}_2$ are equal and opposite, their sum is zero, and the velocity of the particle at $\mathbf{r}$ is simply $\mathbf{A}_2 \times (\mathbf{s}_1 - \mathbf{s}_2)$. This is the same for all particles. Hence two simultaneous equal and opposite angular velocities about parallel axes are equivalent to a velocity of translation of the body as a whole equal to $\mathbf{A}_2 \times (\mathbf{s}_1 - \mathbf{s}_2)$. This is perpendicular to the plane

containing both axes, and of module $p\omega$, where p is the perpendicular distance between the axes and ω the angular speed about either. These results are analogous to those for the resultant of a pair of parallel forces.

Any two simultaneous motions of a rigid body may be similarly compounded. Let the two motions be equivalent to velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ of the point O combined respectively with angular velocities $\mathbf{A}_1$ and $\mathbf{A}_2$ about O. The particle whose position vector relative to O is $\mathbf{r}$ has a velocity equal to the vector sum of the velocities due to each; that is

$$(\mathbf{v}_1 + \mathbf{A}_1 \times \mathbf{r}) + (\mathbf{v}_2 + \mathbf{A}_2 \times \mathbf{r}) = (\mathbf{v}_1 + \mathbf{v}_2) + (\mathbf{A}_1 + \mathbf{A}_2) \times \mathbf{r},$$

showing that the resultant motion is equivalent to a velocity $\mathbf{v}_1 + \mathbf{v}_2$ of the point O combined with an angular velocity $\mathbf{A}_1 + \mathbf{A}_2$ about it. The first of the two simultaneous motions $\mathbf{v}_1$, $\mathbf{A}_1$ is equivalent to a screw whose invariants are $\Gamma_1 = \mathbf{v}_1 \cdot \mathbf{A}_1$ and $\mathbf{A}_1{}^2$, and whose pitch p_1 is their ratio. Similarly for Γ_2 and p_2. The invariants of the combined motion are

$$\Gamma = (\mathbf{v}_1 + \mathbf{v}_2) \cdot (\mathbf{A}_1 + \mathbf{A}_2) = \Gamma_1 + \Gamma_2 + \mathbf{v}_1 \cdot \mathbf{A}_2 + \mathbf{v}_2 \cdot \mathbf{A}_1$$

and
$$(\mathbf{A}_1 + \mathbf{A}_2)^2 = \mathbf{A}_1{}^2 + \mathbf{A}_2{}^2 + 2\mathbf{A}_1 \cdot \mathbf{A}_2,$$

and its pitch p is the ratio of these invariants.

Dynamics of a Rigid Body.

90. Angular momentum. Consider first the case of a *body moving about a fixed point O*. Its motion at any instant consists of an angular velocity $\mathbf{A}$ about this point. If $\mathbf{r}$ is the instantaneous position vector of the particle m relative to O, the velocity of this particle is $\mathbf{A} \times \mathbf{r}$, and the A.M. of the body about O is

$$\begin{aligned} \mathbf{H} &= \Sigma \mathbf{r} \times m\mathbf{v} = \Sigma \mathbf{r} \times m(\mathbf{A} \times \mathbf{r}) \\ &= \Sigma m r^2 \mathbf{A} - \Sigma m \mathbf{r} \cdot \mathbf{A} \mathbf{r}. \quad \text{.............................(12)} \end{aligned}$$

Referred to rectangular coordinate axes through the point O in the directions of $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, let the vectors $\mathbf{r}$, $\mathbf{A}$ have the values

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k},$$
$$\mathbf{A} = \omega_1\mathbf{i} + \omega_2\mathbf{j} + \omega_3\mathbf{k},$$

so that ω_1, ω_2, ω_3 are the angular velocities (or speeds) of the body about the coordinate axes, and x, y, z the coordinates of

the particle m referred to these axes. Substituting these values of $\mathbf{r}$, $\mathbf{A}$ in (12), we find

$$\begin{aligned}\mathbf{H} = \Sigma m\{(y^2+z^2)\omega_1 - xy\omega_2 - xz\omega_3\}\mathbf{i} \\ + \Sigma m\{(z^2+x^2)\omega_2 - yz\omega_3 - yx\omega_1\}\mathbf{j} \\ + \Sigma m\{(x^2+y^2)\omega_3 - zx\omega_1 - zy\omega_2\}\mathbf{k}.\end{aligned}$$

The values of the sums $\Sigma m(y^2+z^2)$, $\Sigma m(z^2+x^2)$, $\Sigma m(x^2+y^2)$ and Σmyz, Σmzx, Σmxy, which are denoted by A, B, C and D, E, F respectively, depend on the distribution of mass relative to the coordinate axes, and are constant only if these axes are fixed in the body. The quantities determined by A, B, C are called the *moments of inertia* of the body about the axes of x, y, z respectively. Those determined by D, E, F are the *products of inertia* relative to the axes of y and z, z and x, x and y respectively. Each moment of inertia involves only one axis. For $A = \Sigma mp^2$ where p is the perpendicular distance of the particle m from the x-axis.

In terms of these moments and products of inertia the A.M. of the body about O may be written

$$\mathbf{H} = h_1\mathbf{i} + h_2\mathbf{j} + h_3\mathbf{k}, \quad \text{.............................(13)}$$

where

$$\left.\begin{aligned}h_1 = A\omega_1 - F\omega_2 - E\omega_3, \\ h_2 = B\omega_2 - D\omega_3 - F\omega_1, \\ h_3 = C\omega_3 - E\omega_1 - D\omega_2.\end{aligned}\right\} \quad \text{.......................(14)}$$

The quantities h_1, h_2, h_3 are the angular momenta of the body about the coordinate axes.

Suppose next that *no point of the body is fixed*, and that we require the A.M. about a given point O. With O as origin let $\bar{\mathbf{r}}$ be the position vector of the c.m. and $\bar{\mathbf{v}} = \dot{\bar{\mathbf{r}}}$ its velocity. Also let $\mathbf{r}'$, $\mathbf{v}'$ be the position vector and velocity of the particle m relative to the c.m. Then, by Art. 84, the A.M. of the body about O is

$$\mathbf{H} = \bar{\mathbf{r}} \times M\bar{\mathbf{v}} + \Sigma\mathbf{r}' \times m\mathbf{v}'. \quad \text{..........................(15)}$$

The second term is the A.M. of the body about the c.m., calculated as though that point were at rest. Its value is therefore given by (13) if we use moments and products of inertia relative to axes through the c.m.

91. Principal axes of inertia. We shall now show that for each point of the rigid body there is one, and in general only one set of mutually perpendicular axes relative to which the products

of inertia vanish. These are called the *principal axes* of inertia at that point.

Let O be the given point of the body, and let us endeavour to find an axis through O such that an angular velocity $\mathbf{A}$ about it gives an A.M. $\mathbf{H}$ about O in the same direction as $\mathbf{A}$. For such an axis we must have

$$\frac{A\omega_1 - F\omega_2 - E\omega_3}{\omega_1} = \frac{B\omega_2 - D\omega_3 - F\omega_1}{\omega_2} = \frac{C\omega_3 - E\omega_1 - D\omega_2}{\omega_3} = \lambda \text{ (say)},$$

or, clearing fractions,

$$\left.\begin{aligned} (A-\lambda)\omega_1 - F\omega_2 - E\omega_3 &= 0, \\ -F\omega_1 + (B-\lambda)\omega_2 - D\omega_3 &= 0, \\ -E\omega_1 - D\omega_2 + (C-\lambda)\omega_3 &= 0. \end{aligned}\right\} \quad \text{......................(i)}$$

Eliminating ω_1, ω_2, ω_3, we have

$$\begin{vmatrix} A-\lambda & -F & -E \\ -F & B-\lambda & -D \\ -E & -D & C-\lambda \end{vmatrix} = 0.$$

This is a cubic equation in λ, and therefore has one real root λ_1. With this value of λ any two of the equations (i) determine the ratios $\omega_1 : \omega_2 : \omega_3$, giving the direction of an axis which possesses the required property. Let this be chosen as the x-axis. Then an angular velocity ω_1 about it gives an A.M. in the same direction. Hence by (14)

$$0 = h_2 = -F\omega_1 \quad \text{and} \quad 0 = h_3 = -E\omega_1,$$

showing that $E = F = 0$. Thus both the products of inertia involving the x-axis vanish.

Try now to find an axis perpendicular to this one, and also possessing the required property that $\mathbf{H}$ is parallel to $\mathbf{A}$. Then for an angular velocity about such an axis $\omega_1 = 0$; and since $E = F = 0$ the condition for parallelism of $\mathbf{A}$ and $\mathbf{H}$ is

$$\frac{B\omega_2 - D\omega_3}{\omega_2} = \frac{C\omega_3 - D\omega_2}{\omega_3} = \mu \text{ (say)}.$$

Hence

$$\left.\begin{aligned} (B-\mu)\omega_2 - D\omega_3 &= 0, \\ -D\omega_2 + (C-\mu)\omega_3 &= 0. \end{aligned}\right\} \quad \text{..........................(ii)}$$

Eliminating $\omega_2 : \omega_3$, we find

$$\mu^2 - \mu(B+C) + (BC - D^2) = 0,$$

which is a quadratic in μ whose roots are easily shown to be real. With a value of μ equal to one of these roots either equation (ii)

determines the ratio $\omega_2 : \omega_3$, which gives an axis having the required property. Let this be taken as the axis of y. Then an angular velocity ω_2 round it produces an A.M. in the same direction. Hence $D = 0$. Since then all the products of inertia vanish, the z-axis is also a principal axis. Thus there are three mutually perpendicular principal axes at the given point O. There is in general only one such set; for the original equation in λ is a cubic, showing that for each point there are only three axes possessing the required property. The moments of inertia about the principal axes are called the *principal moments of inertia* at that point.

For the motion of a body about a fixed point O, if $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ have the directions of the principal axes at that point, and A, B, C are the principal moments of inertia, the formula (13) for the A.M. about O becomes simply

$$\mathbf{H} = A\omega_1\mathbf{i} + B\omega_2\mathbf{j} + C\omega_3\mathbf{k}. \qquad \text{(16)}$$

92. Kinetic Energy. The kinetic energy of a system of particles is defined as the sum of the kinetic energies of the separate particles.

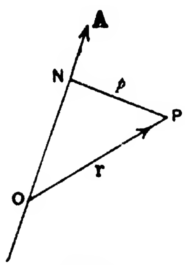

FIG. 58.

Consider a *rigid body moving about a fixed point O* with an angular velocity $\mathbf{A}$ whose module is ω. If $\mathbf{r}$ is the position vector of the particle m relative to O, its velocity is $\mathbf{A}\times\mathbf{r}$, and the kinetic energy of the body is

$$T = \tfrac{1}{2}\Sigma m(\mathbf{A}\times\mathbf{r})^2. \qquad \text{(17)}$$

If p is the perpendicular distance of the particle m from the instantaneous axis, and I the moment of inertia of the body about that axis, $I = \Sigma mp^2$, and the equation (17) may be written

$$T = \tfrac{1}{2}\Sigma mp^2\omega^2 = \tfrac{1}{2}I\omega^2, \qquad \text{(18)}$$

which is analogous to the formula $\tfrac{1}{2}mv^2$ for the kinetic energy of a particle.

It is also worth noticing that if $\mathbf{H}$ is the A.M. of the body about the fixed point O,

$$\begin{aligned}\tfrac{1}{2}\mathbf{A}\cdot\mathbf{H} &= \tfrac{1}{2}\mathbf{A}\cdot\Sigma m\mathbf{r}\times(\mathbf{A}\times\mathbf{r})\\ &= \tfrac{1}{2}\Sigma m(\mathbf{A}\times\mathbf{r})\cdot(\mathbf{A}\times\mathbf{r}) = \tfrac{1}{2}\Sigma m\mathbf{v}^2\\ &= T, \qquad \text{(19)}\end{aligned}$$

a simple relation between the A.M. and the kinetic energy.

Introducing again the same rectangular coordinate axes through O as in Art. 90, let l, m, n be the direction cosines of the instantaneous axis of rotation, so that

$$\mathbf{A} = \omega(l\mathbf{i} + m\mathbf{j} + n\mathbf{k}).$$

Then the expression (17) for the kinetic energy of the body may be written

$$\begin{aligned} T &= \tfrac{1}{2}\Sigma m\{\mathbf{A}^2\mathbf{r}^2 - (\mathbf{A}\cdot\mathbf{r})^2\} \\ &= \tfrac{1}{2}\Sigma m\{(l^2+m^2+n^2)(x^2+y^2+z^2) - (lx+my+nz)^2\}\omega^2 \\ &= \tfrac{1}{2}\{Al^2 + Bm^2 + Cn^2 - 2Dmn - 2Enl - 2Flm\}\omega^2. \end{aligned}$$

Comparing this with the value $\frac{1}{2}I\omega^2$ found above, we have

$$I = Al^2 + Bm^2 + Cn^2 - 2Dmn - 2Enl - 2Flm,$$

a formula which gives the value of the moment of inertia about any axis through a point O, in terms of its direction cosines relative to a set of rectangular axes through O, and the moments and products of inertia relative to these axes.

Suppose now that the body has *no point fixed*. Let $\bar{\mathbf{v}}$ be the velocity of its c.m., and $\mathbf{r}'$ the position vector of the particle m relative to the c.m. The velocity of this particle is then $\bar{\mathbf{v}} + \mathbf{A}\times\mathbf{r}'$, and the kinetic energy of the body is

$$\begin{aligned} T &= \tfrac{1}{2}\Sigma m(\bar{\mathbf{v}} + \mathbf{A}\times\mathbf{r}')^2 \\ &= \tfrac{1}{2}\Sigma m\bar{\mathbf{v}}^2 + \tfrac{1}{2}\Sigma m(\mathbf{A}\times\mathbf{r}')^2 + \bar{\mathbf{v}}\cdot\mathbf{A}\times\Sigma m\mathbf{r}'. \end{aligned}$$

The last term vanishes because $\Sigma m\mathbf{r}' = 0$. Also $\mathbf{A}\times\mathbf{r}'$ is equal to the velocity $\mathbf{v}'$ of the particle m relative to the c.m. Thus

$$T = \tfrac{1}{2}M\bar{\mathbf{v}}^2 + \tfrac{1}{2}\Sigma m\mathbf{v}'^2, \qquad \text{.............................(20)}$$

showing that *the kinetic energy can be expressed as the sum of two parts, one of which represents the kinetic energy of translation of the body as a whole with a velocity equal to that of the c.m., while the other represents the kinetic energy of rotation about the c.m. regarded as a fixed point.*

93. Principle of Energy. It was shown in Art. 75 that the rate of increase of the kinetic energy of a particle is equal to the activity of the resultant force on the particle. Hence, summing for all the particles of the body, we have the rate of increase of the kinetic energy of the body equal to the sum of the activities of all the forces on all its particles. But the internal action

between two particles consists of a pair of equal and opposite forces along the line joining them. And since the distance between two particles of a rigid body remains unaltered, the velocities of the two particles have equal resolutes along the line joining them. Hence the activity of the pair of forces representing their mutual action is zero. The same is true for every pair of forces in the internal action. We need therefore only consider the activity of the external forces on the body; and it follows from the above that *the rate of increase of the kinetic energy of a rigid body is equal to the activity of the external forces*. Hence the increase in the kinetic energy during any finite interval of time is equal to the total work done on the body by external forces during that interval.

94. Moving axes or frame of reference. We have spoken of a constant vector as one whose length and direction remain unchanged. But its direction must be expressed relative to some frame of reference; and this frame will be in motion relative to many other frames that might be used for reference. Thus directions that are constant relative to the former are variable relative to the latter. For practical terrestrial purposes we choose the earth and bodies rigidly attached to it as our standard frame, and both think and speak of it as a fixed frame of reference. We know, however, that it is moving relatively to similar frames belonging to the sun and other heavenly bodies.

Consider two frames of reference, S_1 and S_2, the former of which we may *think* of as fixed, while the latter is moving relatively to it. It will be sufficient to consider a motion of rotation of S_2 with angular velocity $\mathbf{A}$ about a point O fixed in S_1. This point is thus a common fixed point for both frames. Let $\mathbf{r}$ be a variable vector, say the position vector relative to O of some moving point. It is required to determine the relation between its rates of change relative to the two frames. Draw a figure from the point of view of the frame S_1. If in a short interval δt the vector $\mathbf{r}$ changes from $\overrightarrow{OP}$ to $\overrightarrow{OR}$, then $\overrightarrow{PR}$ is the increment $(\delta\mathbf{r})_1$ relative to the frame S_1. Now, during this interval the point of the frame S_2 that was initially at P has moved to Q, where $\overrightarrow{PQ} = \delta t\mathbf{A}\times\mathbf{r}$, so that $\overrightarrow{QR}$ is the increment $(\delta\mathbf{r})_2$ of $\mathbf{r}$ relative to the frame S_2

But $$\overrightarrow{PR} = \overrightarrow{PQ} + \overrightarrow{QR},$$

and therefore $$(\delta\mathbf{r})_1 = \delta t\mathbf{A}\times\mathbf{r} + (\delta\mathbf{r})_2.$$

Dividing by δt, we find the average rate of change during that interval; and proceeding to the limit as δt tends to zero, we obtain the required relation

$$\left(\frac{d\mathbf{r}}{dt}\right)_1 = \left(\frac{d\mathbf{r}}{dt}\right)_2 + \mathbf{A}\times\mathbf{r}, \quad \text{......(21)}$$

the suffix denoting the space relative to which the rate of change is considered.

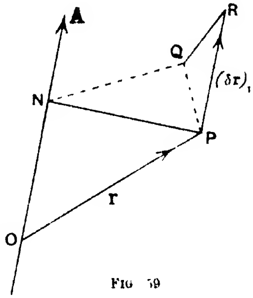

FIG. 59

In this formula $\mathbf{A}$ denotes the angular velocity of S_2 relative to S_1. But the formula is a reciprocal one. For we may regard S_2 as fixed, the angular velocity of S_1 relative to it being $\mathbf{A}' = -\mathbf{A}$, and the above formula gives

$$\left(\frac{d\mathbf{r}}{dt}\right)_2 = \left(\frac{d\mathbf{r}}{dt}\right)_1 + \mathbf{A}'\times\mathbf{r},$$

as we should expect. All motion is relative. Any frame of reference may be regarded as fixed, and the motion of any other expressed relative to it.

95.* Coriolis' theorem. In the equation (21), first take $\mathbf{r}$ as the position vector of a moving point P relative to O. Then its velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ relative to the two frames are connected by

$$\mathbf{v}_1 = \left(\frac{d\mathbf{r}}{dt}\right)_1 = \left(\frac{d\mathbf{r}}{dt}\right)_2 + \mathbf{A}\times\mathbf{r} = \mathbf{v}_2 + \mathbf{A}\times\mathbf{r}. \quad \text{..........(22)}$$

Next apply the formula (21) to the vector $\mathbf{v}_1$, and obtain the acceleration of P relative to S_1 as

$$\begin{aligned}\mathbf{a}_1 = \left(\frac{d\mathbf{v}_1}{dt}\right)_1 &= \left(\frac{d\mathbf{v}_1}{dt}\right)_2 + \mathbf{A}\times\mathbf{v}_1\\ &= \left(\frac{d}{dt}[\mathbf{v}_2 + \mathbf{A}\times\mathbf{r}]\right)_2 + \mathbf{A}\times(\mathbf{v}_2 + \mathbf{A}\times\mathbf{r})\\ &= \left(\frac{d\mathbf{v}_2}{dt}\right)_2 + \left(\frac{d\mathbf{A}}{dt}\right)_2\times\mathbf{r} + \mathbf{A}\times\left(\frac{d\mathbf{r}}{dt}\right)_2 + \mathbf{A}\times\mathbf{v}_2 + \mathbf{A}\times(\mathbf{A}\times\mathbf{r}).\end{aligned}$$

But $$\left(\frac{d\mathbf{A}}{dt}\right)_1 = \left(\frac{d\mathbf{A}}{dt}\right)_2 + \mathbf{A}\times\mathbf{A} = \left(\frac{d\mathbf{A}}{dt}\right)_2,$$

and may therefore be written simply $\dfrac{d\mathbf{A}}{dt}$. The above value for $\mathbf{a}_1$ is therefore

$$\mathbf{a}_1 = \mathbf{a}_2 + 2\mathbf{A}\times\mathbf{v}_2 + \frac{d\mathbf{A}}{dt}\times\mathbf{r} + \mathbf{A}\times(\mathbf{A}\times\mathbf{r}), \quad\text{..................(23)}$$

which is a theorem due to Coriolis. The first two terms depend upon the motion of P relative to S_2; while the other two give the acceleration of a point fixed in S_2 and instantaneously coincident with P. The formula is a reciprocal one; for the suffixes 1 and 2 may be interchanged, provided the sign of $\mathbf{A}$ is changed at the same time.

96. Euler's dynamical equations. Consider a rigid body turning about a fixed point O. We have already seen that, if $\mathbf{H}$ is the A.M. of the body about O, and $\mathbf{L}$ the torque of the external forces about the same point,

$$\frac{d\mathbf{H}}{dt} = \mathbf{L}.$$

This equation was formed expressing the motion of the body relative to some frame S_1 independent of the body, and regarded as fixed. Relative to this frame the body has an angular velocity about O.

$$\mathbf{A} = \omega_1\mathbf{i} + \omega_2\mathbf{j} + \omega_3\mathbf{k}$$

It is, however, frequently convenient to make use of a moving frame S_2 fixed in the body, for expressing the changes of its motion. Such a frame is specified by the principal axes of inertia of the body at O, which are taken as coordinate axes for the (moving) frame S_2. We now consider $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ as remaining parallel to these principal axes, and therefore constant unit vectors relative to the frame S_2. The A.M. of the body about O is, by (16),

$$\mathbf{H} = A\omega_1\mathbf{i} + B\omega_2\mathbf{j} + C\omega_3\mathbf{k},$$

where A, B, C are the principal moments of inertia at O; and the torque of the external forces about O is

$$\mathbf{L} = L_1\mathbf{i} + L_2\mathbf{j} + L_3\mathbf{k},$$

where L_1, L_2, L_3 are their (scalar) moments about the principal axes. Applying (21) to the vector $\mathbf{H}$, we have

$$\mathbf{L} = \left(\frac{d\mathbf{H}}{dt}\right)_1 = \left(\frac{d\mathbf{H}}{dt}\right)_2 + \mathbf{A}\times\mathbf{H}. \quad\text{.........................(24)}$$

This vector equation is equivalent to Euler's scalar equations. For on substituting the above values of **L**, **H** and **A**, remembering that A, B, C and **i**, **j**, **k** are constant relative to S_2, we find an equation which is equivalent to the three scalar equations

$$\left.\begin{aligned} A\frac{d\omega_1}{dt} - (B-C)\omega_2\omega_3 = L_1, \\ B\frac{d\omega_2}{dt} - (C-A)\omega_3\omega_1 = L_2, \\ C\frac{d\omega_3}{dt} - (A-B)\omega_1\omega_2 = L_3. \end{aligned}\right\} \quad \text{.....................(25)}$$

These are Euler's dynamical equations for the motion of a rigid body about a fixed point, referred to axes fixed in the body, and coinciding with the principal axes of inertia at the fixed point.

Examples.

(1) *Prove that, in the above problem, if T is the kinetic energy of the body,*

$$\frac{dT}{dt} = \mathbf{A}\cdot\mathbf{L}.$$

On forming the scalar product of both members of (24) with **A**, the last term vanishes, showing that

$$\begin{aligned} \mathbf{A}\cdot\mathbf{L} &= \mathbf{A}\cdot\left(\frac{d\mathbf{H}}{dt}\right)_1 = \mathbf{A}\cdot\left(\frac{d\mathbf{H}}{dt}\right)_2 \\ &= A\omega_1\frac{d\omega_1}{dt} + B\omega_2\frac{d\omega_2}{dt} + C\omega_3\frac{d\omega_3}{dt} \\ &= \left(\frac{d\mathbf{A}}{dt}\right)_2\cdot\mathbf{H} = \left(\frac{d\mathbf{A}}{dt}\right)_1\cdot\mathbf{H}. \end{aligned}$$

Hence

$$\begin{aligned} \mathbf{A}\cdot\mathbf{L} &= \tfrac{1}{2}\mathbf{A}\cdot\left(\frac{d\mathbf{H}}{dt}\right)_1 + \tfrac{1}{2}\left(\frac{d\mathbf{A}}{dt}\right)_1\cdot\mathbf{H} \\ &= \frac{1}{2}\frac{d}{dt}(\mathbf{A}\cdot\mathbf{H}) = \frac{dT}{dt}, \end{aligned}$$

by (19). The expression $\mathbf{A}\cdot\mathbf{L}$ therefore measures the activity of the torque **L** acting on the body.

(2) *If, in the same problem, the kinetic energy is proportional to the square of the angular momentum, prove that the plane of* **H** *and* **L** *is perpendicular to that of* **H** *and* **A**.

The kinetic energy $T = k\mathbf{H}^2$, where k is constant. The vector $\mathbf{H}\times\mathbf{L}$ is normal to the plane of **H** and **L**; and $\mathbf{H}\times\mathbf{A}$ is normal to that of **H** and **A**. These are perpendicular if

$$(\mathbf{H}\times\mathbf{L})\cdot(\mathbf{H}\times\mathbf{A}) = 0.$$

But the first member when expanded is

$$\mathbf{H}^2\mathbf{L}\cdot\mathbf{A} - \mathbf{H}\cdot\mathbf{AL}\cdot\mathbf{H} = \mathbf{H}^2\frac{dT}{dt} - 2T\frac{d\mathbf{H}}{dt}\cdot\mathbf{H}$$

$$= \mathbf{H}^2\frac{d}{dt}(k\mathbf{H}^2) - k\mathbf{H}^2\frac{d}{dt}(\mathbf{H}^2),$$

which is obviously zero. Hence the result.

EXERCISES ON CHAPTER VII.

1. A homogeneous sphere rolls without slipping on a fixed rough plane under the action of forces whose resultant passes through the centre of the sphere. Show that the motion of the sphere is the same as if the plane were smooth, and all the forces were reduced to five-sevenths of their former value.

2. A cube is rotating with angular velocity ω about a diagonal, when suddenly one of its edges which does not meet the diagonal becomes fixed. Show that the ensuing angular velocity about this edge is $\frac{\omega}{12}\sqrt{3}$.

3. An inelastic cube, sliding down a plane inclined at α to the horizontal, strikes symmetrically a small fixed nail with speed V. If it tumbles over the nail and goes on sliding down the plane, show that the value of V^2 is not less than $16ga(\sqrt{2} - \cos\alpha - \sin\alpha)/3$, where $2a$ is the length of an edge of the cube.

4. A rod moves with its extremities on two intersecting lines. Find the direction of motion of any point. If two lines do not intersect but are at right angles, examine whether the motion can be represented by an angular velocity only.

5. Prove that a straight line through the c.m. G of a body, which is a principal axis of inertia at G, is a principal axis at any point.

6. A rigid body, hinged at O to a fixed point, is set rotating about a principal axis at O. If it is acted on by no forces but those at the hinge, show that it will continue to rotate with constant angular velocity about the same axis.

If O is the c.m. of the body, show that there is zero action on the hinge; and hence that a principal axis at the c.m. is an axis of free rotation.

7. A fly-wheel of mass M, concentrated at the rim of radius a, is rotating with angular velocity ω about a fixed axis through its centre inclined at an angle θ to the axis of the wheel. Show that the constraint due to the fixed axis of rotation is equivalent to a couple

whose scalar moment is $\frac{1}{2}M\omega^2a^2 \sin\theta\cos\theta$, and whose plane contains the axis of the wheel and the axis of rotation.

8. Determine the screw motion which is equivalent to the two screws $\mathbf{v}$, $\mathbf{A}$ and $\mathbf{v}'$, $\mathbf{A}'$ whose axes are given; and show that its invariant Γ is

$$(\mathbf{v}+\mathbf{v}')\cdot(\mathbf{A}+\mathbf{A}')+AA'M,$$

where M is the mutual moment of the two axes. Also prove that the axis of the resultant screw intersects at right angles the common perpendicular to the axes of the two given screws.

9. Prove that the points, whose position vectors relative to a given origin are equal to the velocity vectors of the particles of a rigid body, all lie in a plane.

10. Show that any motion of a rigid body may be represented by two angular velocities $\mathbf{A}$ and $\mathbf{A}'$, about axes one of which may be chosen arbitrarily. Also that the common perpendicular to the two axes intersects perpendicularly the axis of the resultant screw. (The axes of $\mathbf{A}$, $\mathbf{A}'$ are called *conjugate axes.*)

11. If one conjugate axis of an instantaneous motion is perpendicular to the axis of the screw, the other meets this axis: and conversely.

12. A body possesses simultaneously twelve equal angular velocities about axes forming the edges of a cube, those about parallel axes being in the same sense. Prove that the axis of the resultant motion is a diagonal of the cube.

13. A body possesses simultaneous angular velocities about the sides of a skew polygon taken in order, their magnitudes being proportional to the lengths of the corresponding sides. Show that every point of the body has the same velocity.

14. If four simultaneous angular velocities are equivalent to zero motion in the body, show that the invariant of any two is equal to that of the other two. Also that the invariant of any three is zero.

15. The coordinates of a moving point P are x, y relative to rectangular axes traced on a plane lamina which is rotating about the origin, in its own plane, with variable angular velocity ω. Prove that the acceleration of P has resolutes, in the directions occupied by the moving axes at any instant, given by

$$\ddot{x}-y\dot{\omega}-x\omega^2-2\omega\dot{y} \quad \text{and} \quad \ddot{y}+x\dot{\omega}-y\omega^2+2\omega\dot{x}.$$

16. A particle B is moving on a smooth plane curve which is rotating in its own plane with constant angular velocity ω about a fixed origin O. If P, Q are the tangential and normal resolutes of the force on the particle, R the reaction of the curve, and ϕ the

angle which OB makes with the tangent, show that the equations of motion may be put in the form

$$m\left[\frac{d}{ds}(\tfrac{1}{2}v^2) - \omega^2 r\frac{dr}{ds}\right] = P,$$

$$m(\kappa v^2 + 2\omega v + \omega^2 r\sin\phi) = R + Q,$$

s being the length of the arc measured from a fixed point on the curve.

17. A circular wire is constrained to turn round a vertical tangent with a uniform angular velocity ω. A smooth heavy bead, starting from the highest point without any velocity relative to the wire, descends under the action of gravity. Find the velocity and the reaction in any position.

18. A smooth helical wire is constrained to turn about its axis, which is vertical, with uniform angular velocity ω. Find the motion of a particle descending on it under the action of gravity.

19. A heavy particle is moving in a smooth surface of revolution whose axis Oz is vertical, and vertex downward. If z, r, ϕ are the cylindrical coordinates of the particle, s the length of the meridian arc from the vertex, R the reaction of the surface and $\omega = \dot{\phi}$, prove the equations of motion in the form

$$m\left(\ddot{s} - \omega^2 r\frac{dr}{ds}\right) = -mg\frac{dz}{ds},$$

$$m\left[\kappa v^2 + \omega^2 r\frac{dz}{ds}\right] = R - mg\frac{dr}{ds},$$

$$m\frac{1}{r}\frac{d}{dt}(\omega r^2) = 0.$$

20. The position of a moving point is given in spherical polar coordinates r, θ, ϕ. Find the resolutes of its acceleration in the radial direction, perpendicular to the meridian plane and in the meridian plane.

21. Hence show that the equations of motion of a heavy particle, tied to a fixed point by a light inextensible string of length l, are

$$m(l\dot{\theta}^2 + l\sin^2\theta\dot{\phi}^2) = T - mg\cos\theta,$$

$$l\ddot{\theta} - l\cos\theta\sin\theta\dot{\phi}^2 = -g\sin\theta,$$

$$\frac{1}{\sin\theta}\frac{d}{dt}(\sin^2\theta\dot{\phi}) = 0.$$

22. A solid cubical body is in motion, under no external forces, about a fixed corner. If ω_1, ω_2, ω_3 are the angular velocities about the three edges meeting at the fixed point, prove that $\omega_1 + \omega_2 + \omega_3$ and $\omega_1^2 + \omega_2^2 + \omega_3^2$ are both constant.

23. A body turning about a fixed point is acted on by forces which tend to produce rotation about an axis perpendicular to **A**. Show that the angular velocity cannot be uniform unless two of the principal moments of inertia at the fixed points are equal.

24. A rigid body is acted on by a force **F** per unit mass, varying from point to point. If, relative to a fixed origin O, **r** is the position vector of the point P where the density is μ, $\bar{\mathbf{v}}$ the velocity of the c.m., and **H** the A.M. about O, show that

$$M\frac{d\bar{\mathbf{v}}}{dt}=\int \mu\mathbf{F}\,dv,$$

$$\frac{d\mathbf{H}}{dt}=\int \mu\mathbf{r}\times\mathbf{F}\,dv,$$

where dv is the element of volume at P.

25. A symmetrical body, such as a top, is rotating about a fixed point O on the axis of symmetry; and the external forces have zero moment about that axis ($L_3=0$). The principal moments of inertia at O are A, A, C. Show from Euler's dynamical equations that ω_3 remains constant. Then, by considering the velocity **v** and the acceleration **a** of the point P on the axis of symmetry whose position vector is **k** relative to O, show that

$$A\mathbf{a}=L_2\mathbf{i}-L_1\mathbf{j}+C\omega_3\mathbf{k}\times\mathbf{v}-A(\omega_1^2+\omega_2^2)\mathbf{k}.$$

Hence prove that P moves as a particle of mass A smoothly constrained to the surface of a unit sphere (centre O) and acted on by tangential forces $C\omega_3\mathbf{k}\times\mathbf{v}$ and **F**, the latter having the same torque about O as the external forces.

26. If, in the previous exercise, the only external force is the weight of the body, and the c.m. is at a distance h from O, show that for a steady precession of the axis at an angle θ to the vertical, the angular velocity Ω of precession is given by

$$\Omega=(C\omega_3-\sqrt{C^2\omega_3^2-4AMgh\cos\theta})/2A\cos\theta,$$

where M is the mass of the body.

CHAPTER VIII.

STATICS OF A RIGID BODY.

97. Conditions of equilibrium. The equations of motion for a rigid body, as found in the previous chapter, are equivalent to

$$\frac{d\mathbf{M}}{dt} = \Sigma\mathbf{F},$$

$$\frac{d\mathbf{H}}{dt} = \Sigma\mathbf{r}\times\mathbf{F},$$

where $\mathbf{M}$ is the linear momentum of the body, $\mathbf{H}$ its A.M. about the c.m. (Art. 85), $\Sigma\mathbf{F}$ the vector sum of the external forces, and $\Sigma\mathbf{r}\times\mathbf{F}$ the torque of these forces about the c.m. If the body remains at rest under the action of the forces, $\mathbf{M}$ and $\mathbf{H}$ are permanently zero. Thus

$$\Sigma\mathbf{F} = 0 \quad \text{and} \quad \Sigma\mathbf{r}\times\mathbf{F} = 0 \quad \text{......................(1)}$$

are *necessary* conditions of equilibrium for the body.

But they are also *sufficient* conditions, provided the body is initially at rest. For, if they are satisfied, $\mathbf{M}$ and $\mathbf{H}$ are constant, and therefore remain permanently zero. But

$$\mathbf{M} = M\bar{\mathbf{v}},$$

and
$$\mathbf{H} = A\omega_1\mathbf{i} + B\omega_2\mathbf{j} + C\omega_3\mathbf{k},$$

the principal axes at the c.m. being chosen for reference. Hence $\bar{\mathbf{v}}$, ω_1, ω_2, ω_3 must all remain equal to zero, and the body is therefore in equilibrium. The conditions (1) are therefore sufficient conditions of equilibrium for the body.

Suppose we take moments about any other point P whose position vector relative to the c.m. is $\mathbf{r}'$. Then the torque of the external forces about P is

$$\Sigma(\mathbf{r} - \mathbf{r}')\times\mathbf{F} = \Sigma\mathbf{r}\times\mathbf{F} - \mathbf{r}'\times\Sigma\mathbf{F},$$

which is zero in virtue of (1). Thus the condition of zero torque is satisfied for any point. Conversely, if $\Sigma\mathbf{F}=0$ and the torque about any one point is zero, it is zero about all points. In the conditions (1) therefore, the point chosen as origin of moments may be any whatever.

The conditions of equilibrium just found are equivalent to six **scalar conditions.** For if, choosing rectangular axes through the origin of moments, we write

$$\mathbf{F}=X\mathbf{i}+Y\mathbf{j}+Z\mathbf{k},$$
$$\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k},$$

the equations (1) become

$$\Sigma(X\mathbf{i}+Y\mathbf{j}+Z\mathbf{k})=0,$$

and $$\Sigma(yZ-zY)\mathbf{i}+\Sigma(zX-xZ)\mathbf{j}+\Sigma(xY-yX)\mathbf{k}=0,$$

which are equivalent to the six scalar conditions

$$\left.\begin{array}{c}\Sigma X=0,\quad \Sigma Y=0,\quad \Sigma Z=0,\\ \Sigma(yZ-zY)=0,\\ \Sigma(zX-xZ)=0,\\ \Sigma(xY-yX)=0,\end{array}\right\}\qquad\qquad(2)$$

that is, the sum of the resolved parts of the forces must vanish for each of the coordinate axes, and the moment of the external forces about each of them must also vanish.

There is no need to choose rectangular axes as we have done, but these are generally more convenient than oblique. And, with regard to moments about a line, it is worth noticing that if $\Sigma\mathbf{F}=0$ the moment is the same for all parallel lines. For the moment about a line parallel to $\hat{\mathbf{a}}$, drawn through the point P whose position vector is $\mathbf{r}'$, is

$$\hat{\mathbf{a}}\cdot\Sigma(\mathbf{r}-\mathbf{r}')\times\mathbf{F}=\hat{\mathbf{a}}\cdot\Sigma\mathbf{r}\times\mathbf{F}-\hat{\mathbf{a}}\cdot\mathbf{r}'\times\Sigma\mathbf{F}$$
$$=\hat{\mathbf{a}}\cdot\Sigma\mathbf{r}\times\mathbf{F},$$

which is independent of $\mathbf{r}'$, and therefore the same for all axes parallel to $\hat{\mathbf{a}}$.

98. Equivalent systems of forces. From the conditions of equilibrium found in the previous Art. it follows that two systems of forces are statically equivalent in their action on a rigid body

if their vector sums are equal and also their torques about any point. For if one of the systems is reversed, and then both act together on a rigid body, the combined system of forces satisfies the conditions (1), and the body is in equilibrium. Thus the statical effect of a system of forces on a body is completely determined by its vector sum $\Sigma\mathbf{F}$, and its torque $\Sigma\mathbf{r}\times\mathbf{F}$ about a specified point.

The point of application of a force may therefore be shifted to any other point in its line of action; for this does not alter either of the above quantities. This is the principle of the *transmissibility of forces.*

Further, the effect of a system of forces on a rigid body is unaltered by introducing a pair of equal and opposite forces with the same line of action. For the vector sum of such a pair is zero, and also its torque about any point.

99. Parallel forces. Centre of gravity. Given a system of parallel forces $p_1\mathbf{a}$, $p_2\mathbf{a}, \ldots, p_n\mathbf{a}$ acting through the points $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n$ respectively, it is required to find a single force which is their statical equivalent. If there is such a force it must, by the preceding Art., be equal to the vector sum of the separate forces, that is to $(p_1+p_2+\ldots+p_n)\mathbf{a}$. And further, it must have the same torque about the origin. If then it acts through the point $\bar{\mathbf{r}}$ we must have

$$\bar{\mathbf{r}}\times(\Sigma p)\mathbf{a}=\Sigma\mathbf{r}\times p\mathbf{a},$$

or

$$\bar{\mathbf{r}}\times\mathbf{a}=\frac{\Sigma p\mathbf{r}}{\Sigma p}\times\mathbf{a}.$$

Hence

$$\bar{\mathbf{r}}=\frac{\Sigma p\mathbf{r}}{\Sigma p}+t\mathbf{a},$$

where the value of t is arbitrary. Thus the single force, which is equivalent to the system of parallel forces, acts through the point $\Sigma p\mathbf{r}/\Sigma p$, its line of action being parallel to the others, and its magnitude equal to the sum of their magnitudes. This point is independent of the direction of the parallel forces, and is the centroid of the points $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n$ with associated numbers $p_1, p_2, \ldots, p_n$.

In particular, if there are particles of masses $m_1, m_2, \ldots, m_n$ respectively at the above points, their weights constitute a system of parallel forces which may be represented by $m_1\mathbf{a}, m_2\mathbf{a}, \ldots, m_n\mathbf{a}$;

and the single force which is equivalent to these has a line of action passing through the point

$$\bar{\mathbf{r}} = \frac{\Sigma m\mathbf{r}}{\Sigma m}$$

whatever the direction of $\mathbf{a}$. This point is called the *centre of gravity* of the system of particles, and is identical with their centre of mass, as defined in Art. 11.

If the sum $\Sigma p\mathbf{a}$ of the parallel forces is zero, Σp vanishes. If $\Sigma p\mathbf{r}$ is also zero they have zero torque about the origin and are in equilibrium. But if not, their resultant is a zero force acting at an infinite distance from the origin. This case will be considered in the following Art. Here we observe that for a system of particles Σm cannot vanish, so that they always have a c.m. given by the above formula.

100. Couples. Law of composition. Consider a pair of parallel forces, equal in magnitude but opposite in direction and localised in different lines. Such a pair of forces is called a *couple*, and the plane containing the two lines of action is the *plane of the couple*. The vector sum of the forces is zero, and there is no single force which is statically equivalent to the couple.

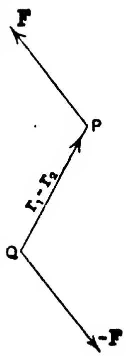

FIG. 60.

Let $\mathbf{F}$ and $-\mathbf{F}$ be the two forces, with lines of action passing through the points P and Q, whose position vectors relative to an origin O are $\mathbf{r}_1$ and $\mathbf{r}_2$ respectively. Then the torque of the couple about O is

$$\mathbf{r}_1 \times \mathbf{F} + \mathbf{r}_2 \times (-\mathbf{F}) = (\mathbf{r}_1 - \mathbf{r}_2) \times \mathbf{F}.$$

But $\mathbf{r}_1 - \mathbf{r}_2$ is the vector $\overrightarrow{QP}$, and the torque is therefore independent of the point O. It may therefore be called simply the *torque of the couple*. Its direction is perpendicular to the plane of the couple; and its magnitude, called the scalar moment, is measured by Fp, where p is the perpendicular distance between the lines of action of the forces.

It follows from Art. 98 that *two couples are statically equivalent if they have the same torque*. For $\Sigma\mathbf{F}$ is zero for both couples

and $\Sigma\mathbf{r}\times\mathbf{F}$ is the same for each, since their torques are equal. Thus for two couples to be equivalent their planes must be parallel, for each is perpendicular to the vector representing the torque. The direction of the forces in the plane is immaterial; but the scalar moment Fp must be the same for both couples, and the sense of the torque also the same. The statical effect of a couple acting on a rigid body is thus completely determined by its torque.

Suppose a body acted on by several couples whose torques are $\mathbf{L}_1, \mathbf{L}_2, \ldots, \mathbf{L}_n$ respectively. Then for the whole system of forces the vector sum is zero, and the torque about any point is

$$\mathbf{L} = \mathbf{L}_1 + \mathbf{L}_2 + \ldots + \mathbf{L}_n.$$

The whole system is therefore equivalent to a single couple whose torque $\mathbf{L}$ is the vector sum of the torques of the separate couples. This is what is meant by saying that *couples are compounded by vector addition of their torques*

101. Poinsot's reduction of a system of forces. Let the body be acted on by forces $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n$ with lines of action passing through the points whose position vectors relative to an origin O are $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n$ respectively. The statical effect of the system is unaltered by introducing at O pairs of equal and opposite forces $\pm\mathbf{F}_1, \pm\mathbf{F}_2, \ldots, \pm\mathbf{F}_n$. The forces $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n$ at O have a resultant

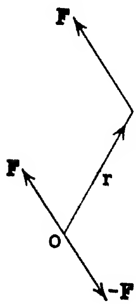

FIG. 61.

$$\mathbf{R} = \Sigma\mathbf{F} \quad \text{.........................(1)}$$

through that point; and the remaining forces on the body constitute couples whose torques are $\mathbf{r}_1\times\mathbf{F}_1, \mathbf{r}_2\times\mathbf{F}_2, \ldots, \mathbf{r}_n\times\mathbf{F}_n$ respectively. These are equivalent to a single couple whose torque is

$$\mathbf{G} = \Sigma\mathbf{r}\times\mathbf{F}. \quad \text{.........................(2)}$$

Thus the original system of forces is equivalent to a single force $\mathbf{R}$ through O, equal to the vector sum of the forces, together with a couple whose torque is equal to the vector sum of the torques of the separate forces about O.

The force $\mathbf{R}$ is the same for all origins; but not so the couple $\mathbf{G}$. For if we take as origin a point O' whose position vector relative

to O is $\mathbf{s}$ (Fig. 55), we find the system equivalent to a force $\mathbf{R}$ through O', together with a couple whose torque is

$$\begin{aligned}\mathbf{G}' &= \Sigma(\mathbf{r}-\mathbf{s})\times\mathbf{F} = \Sigma\mathbf{r}\times\mathbf{F} - \mathbf{s}\times\Sigma\mathbf{F} \\ &= \mathbf{G} - \mathbf{s}\times\mathbf{R}. \qquad\qquad (3)\end{aligned}$$

This relation is, of course, obvious from the fact that a force $\mathbf{R}$ at O' is equivalent to an equal force at O together with a couple of torque $\mathbf{s}\times\mathbf{R}$.

Since $\mathbf{R}$ is the same for all origins, $\mathbf{R}^2$ is an *invariant* of the system. Another invariant is the scalar product of $\mathbf{R}$ and $\mathbf{G}$. For by (3)
$$\mathbf{R}\cdot\mathbf{G}' = \mathbf{R}\cdot\mathbf{G} - [\mathbf{R}\mathbf{s}\mathbf{R}] = \mathbf{R}\cdot\mathbf{G}.$$

We shall denote this second invariant by Γ. It expresses the fact that the scalar moment of the system of forces is the same about all lines parallel to $\mathbf{R}$. These invariants are analogous to those of Art. 88.

We naturally enquire if there is a point for which $\mathbf{G}$ is parallel to $\mathbf{R}$. If O' is such a point, the vector product $\mathbf{R}\times\mathbf{G}'$ must vanish, and therefore
$$\mathbf{R}\times(\mathbf{G}-\mathbf{s}\times\mathbf{R}) = 0,$$
$$\mathbf{R}\times\mathbf{G} - \mathbf{R}^2\mathbf{s} + \mathbf{R}\cdot\mathbf{s}\mathbf{R} = 0,$$

from which it follows that
$$\mathbf{s} = \frac{\mathbf{R}\times\mathbf{G}}{\mathbf{R}^2} + t\mathbf{R},$$

where, by substitution, it is found that t is arbitrary. Thus the locus of points O' possessing the required property is the straight line through $\mathbf{R}\times\mathbf{G}/\mathbf{R}^2$ parallel to $\mathbf{R}$ (cf. Fig. 57, where $\mathbf{A}$ corresponds to $\mathbf{R}$ and $\mathbf{v}$ to $\mathbf{G}$). This straight line is called the **central axis** of the system; and the system of forces has thus been proved equivalent to a force $\mathbf{R}$ along the central axis, together with a couple whose plane is perpendicular to it. Such a force and couple constitute what is called a *wrench*. The line of action of the force is the axis of the wrench. The torque of the couple is clearly the same for all points on the central axis. For if in (3) $\mathbf{s}$ is increased by $t\mathbf{R}$ the value of $\mathbf{G}'$ is unaltered. It is called the *principal torque*. The *pitch* p of the wrench is the ratio of the parallel vectors $\mathbf{G}'$ and $\mathbf{R}$. Thus
$$p = \frac{\mathbf{G}'}{\mathbf{R}} = \frac{\mathbf{G}'\cdot\mathbf{R}}{\mathbf{R}^2} = \frac{\Gamma}{\mathbf{R}^2},$$

and is therefore the ratio of the two invariants of the system. The pitch, as thus defined, is positive if $\mathbf{G}'$ and $\mathbf{R}$ have the same direction, *i.e.* if the wrench is right-handed; negative if the wrench is left-handed. If the system of forces reduces to a couple only, $\mathbf{R}$ is zero and the pitch is infinite. If it reduces to a single force $\mathbf{R}$ at O', the couple $\mathbf{G}'$ is zero and the pitch vanishes.

Examples.

(1) *Forces of magnitudes la, mb, nc act along three non-intersecting edges of a parallelepiped whose lengths are a, b, c respectively. Prove that the invariant Γ of the system is $(mn+nl+lm)V$, where V is the volume of the figure.*

Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be the vectors determined by the edges of the parallelepiped, all directed from one corner, which we take as origin; and let forces $l\mathbf{a}$, $m\mathbf{b}$, $n\mathbf{c}$ act respectively through the origin and the points $\mathbf{c}$ and $(\mathbf{a}+\mathbf{b})$. Then the vector sum of the forces is

$$\mathbf{R}=l\mathbf{a}+m\mathbf{b}+n\mathbf{c},$$

and the torque about the origin is

$$\mathbf{G}=\mathbf{c}\times m\mathbf{b}+(\mathbf{a}+\mathbf{b})\times n\mathbf{c}.$$

Hence the invariant

$$\begin{aligned}\mathbf{R}\cdot\mathbf{G}&=(l\mathbf{a}+m\mathbf{b}+n\mathbf{c})\cdot(n\mathbf{a}\times\mathbf{c}+n\mathbf{b}\times\mathbf{c}+m\mathbf{c}\times\mathbf{b})\\&=(lm+mn+nl)[\mathbf{abc}],\end{aligned}$$

which proves the result.

(2) *Two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ act along non-intersecting lines. Prove that their central axis intersects the common perpendicular to the two lines, and divides it in the ratio $\mathbf{F}_2\cdot(\mathbf{F}_1+\mathbf{F}_2):\mathbf{F}_1\cdot(\mathbf{F}_1+\mathbf{F}_2)$. Also that the scalar moment of the principal couple is $F_1F_2M/\,|\mathbf{F}_1+\mathbf{F}_2|$ where M is the mutual moment of the lines of action of the two forces.*

Let PP' (Fig. 42) be the common perpendicular. The central axis is parallel to the vector sum $\mathbf{F}_1+\mathbf{F}_2=\mathbf{R}$ say, and is therefore at right angles to PP', since each force is so. Further, each of these forces has zero moment about PP'; and therefore the equivalent wrench $\mathbf{R}$, $\mathbf{G}$ has zero moment. But the central axis is perpendicular to PP'; so that $\mathbf{G}$ has zero resolved part along it. Hence the central axis cuts PP', for otherwise the moment of the force $\mathbf{R}$ about this line would not vanish.

Take the point O of intersection of these lines as origin, and let $\overrightarrow{OP}=m\mathbf{k}$ and $\overrightarrow{OP'}=-n\mathbf{k}$, $\mathbf{k}$ being perpendicular to $\mathbf{F}_1$ and $\mathbf{F}_2$. Then the torque of these forces about O is

$$m\mathbf{k}\times\mathbf{F}_1-n\mathbf{k}\times\mathbf{F}_2,$$

and since this is parallel to $\mathbf{R}$,

$$\{\mathbf{k}\times(m\mathbf{F}_1 - n\mathbf{F}_2)\}\times(\mathbf{F}_1 + \mathbf{F}_2) = 0.$$

Expanding this by Art. 44, we find

$$n\mathbf{F}_2\cdot(\mathbf{F}_1 + \mathbf{F}_2)\mathbf{k} - m\mathbf{F}_1\cdot(\mathbf{F}_1 + \mathbf{F}_2)\mathbf{k} = 0,$$

which gives the first result.

Finally equate the values of the invariant for the origins O and P'. Then

$$\begin{aligned}\mathbf{R}\cdot\mathbf{G} &= (m+n)\mathbf{k}\times\mathbf{F}_1\cdot(\mathbf{F}_1 + \mathbf{F}_2)\\ &= (m+n)\mathbf{k}\times\mathbf{F}_1\cdot\mathbf{F}_2\\ &= MF_1F_2,\end{aligned}$$

where M is the mutual moment of the lines. Hence, since $\mathbf{R}$ and $\mathbf{G}$ are parallel,

$$G = \frac{MF_1F_2}{\mathbf{R}},$$

which is the required result.

(3) *Two wrenches* $(\mathbf{F}_1,\ p_1\mathbf{F}_1)$ *and* $(\mathbf{F}_2,\ p_2\mathbf{F}_2)$ *acting on a body have axes whose mutual moment is* M, *and whose shortest distance apart is* $2h$. *Show that the central axis of their resultant wrench intersects perpendicularly the common perpendicular to their axes, at a distance from its middle point equal to*

$$\frac{2h^2(\mathbf{F}_1{}^2 - \mathbf{F}_2{}^2) - MF_1F_2(p_1 - p_2)}{2h(\mathbf{F}_1 + \mathbf{F}_2)^2}.$$

It may be shown exactly as in the previous example that the central axis intersects the common perpendicular PP' (Fig. 42) at right angles. With the point O of intersection as origin, let

$$\overrightarrow{OP} = (h - x)\mathbf{k} \quad \text{and} \quad \overrightarrow{OP'} = -(h + x)\mathbf{k}.$$

The torque about O of the original system is

$$\mathbf{G} = (h - x)\mathbf{k}\times\mathbf{F}_1 - (h + x)\mathbf{k}\times\mathbf{F}_2 + p_1\mathbf{F}_1 + p_2\mathbf{F}_2.$$

And since this is parallel to the central axis

$$\mathbf{G}\times(\mathbf{F}_1 + \mathbf{F}_2) = 0,$$

which gives $\quad x(\mathbf{F}_1 + \mathbf{F}_2)^2\mathbf{k} = h(\mathbf{F}_1{}^2 - \mathbf{F}_2{}^2)\mathbf{k} - (p_1 - p_2)\mathbf{F}_1\times\mathbf{F}_2.$

On forming the scalar product of both sides with $2h\mathbf{k}$ the result follows.

102. Null plane at a point. Let the system of forces acting on the body be equivalent to a force $\mathbf{R}$ through O and a couple of torque $\mathbf{G}$. Then the scalar moment of the system about any straight line through O perpendicular to $\mathbf{G}$ is zero. All such lines lie in the plane through O perpendicular to $\mathbf{G}$, which is

called the *null plane* at the point O, while O is called the *null point* of the plane.

If the point O' lies in the null plane at O, the line OO' is a null line; that is to say the system has zero moment about it. Hence it lies in the null plane at O'. Thus, *if the null plane at O passes through O', the null plane at O' passes through O.*

To find the equation of the null plane at the point $\mathbf{s}$, relative to O as origin, we have only to observe that the couple for that point is

$$\mathbf{G}' = \mathbf{G} - \mathbf{s}\times\mathbf{R},$$

and the equation of the plane through $\mathbf{s}$ perpendicular to $\mathbf{G}'$ is

$$(\mathbf{r} - \mathbf{s})\cdot(\mathbf{G} - \mathbf{s}\times\mathbf{R}) = 0$$

or

$$\mathbf{r}\cdot(\mathbf{G} - \mathbf{s}\times\mathbf{R}) = \mathbf{s}\cdot\mathbf{G}.$$

This is the equation of the null plane at $\mathbf{s}$. It is symmetrical in $\mathbf{r}$ and $\mathbf{s}$; so that if the null plane at $\mathbf{s}$ passes through $\mathbf{r}$, the null plane at $\mathbf{r}$ passes through $\mathbf{s}$.

103. Conjugate forces. *Any system of forces acting on a body is statically equivalent to two forces, of which the line of action of one may be chosen arbitrarily.* Such a pair of forces are called *conjugate forces* of the system.

Take a straight line through an arbitrary point O in any direction, say that of $\hat{\mathbf{a}}$. Let the system of forces be equivalent to a force $\mathbf{R}$ through O and a couple $\mathbf{G}$. We can resolve $\mathbf{R}$ into components $\mathbf{F}_1 = F_1\hat{\mathbf{a}}$ and $\mathbf{F}_2 = \mathbf{R} - \mathbf{F}_1$ in such a way that $\mathbf{F}_2$ is parallel to the plane of the couple $\mathbf{G}$. This only requires

$$0 = \mathbf{F}_2\cdot\mathbf{G} = (\mathbf{R} - F_1\hat{\mathbf{a}})\cdot\mathbf{G},$$

that is

$$F_1 = \frac{\mathbf{R}\cdot\mathbf{G}}{\hat{\mathbf{a}}\cdot\mathbf{G}}.$$

Replace $\mathbf{R}$ by these two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ at O. We can then transform the couple $\mathbf{G}$ to consist of a force $-\mathbf{F}_2$ at O and another $\mathbf{F}_2$ in the null plane at O. The system is then equivalent to $\mathbf{F}_1$ at O parallel to $\hat{\mathbf{a}}$ and $\mathbf{F}_2 = \mathbf{R} - F_1\hat{\mathbf{a}}$ in the null plane at O. Both forces are uniquely determined; and the line of action of $\mathbf{F}_2$ is also determined, for the torque of this force about O must be equal to $\mathbf{G}$. Thus, $\mathbf{G} = \mathbf{r}\times\mathbf{F}_2$, where $\mathbf{r}$ is the position vector of any point on this line of action, referred to O as origin.

The axis of $\mathbf{F}_2$ may also be expressed as the line of intersection of two planes. For since this axis lies in the null plane at O, any point on it must satisfy the equation

$$\mathbf{r}\cdot\mathbf{G} = 0. \quad\text{.......................................(1)}$$

But it also lies on the null plane of any point $a\hat{\mathbf{a}}$ on the axis of $\mathbf{F}_1$; that is, in the plane

$$\mathbf{r}\cdot(\mathbf{G} - a\hat{\mathbf{a}}\times\mathbf{R}) = a\hat{\mathbf{a}}\cdot\mathbf{G}.$$

By subtraction we find that

$$\mathbf{r}\cdot(\mathbf{R}\times\hat{\mathbf{a}}) = \hat{\mathbf{a}}\cdot\mathbf{G} \quad\text{..............................(2)}$$

is a plane through the line of intersection of the other two. Thus the axis of $\mathbf{F}_2$ is the line of intersection of (1) and (2). The lines of action of the conjugate forces $\mathbf{F}_1$ and $\mathbf{F}_2$ are called *conjugate lines* or conjugate axes.

104. Principle of Virtual Work or Virtual Velocities. *If a system of bodies, in equilibrium under any set of forces, is supposed started with any finite motion consistent with the connections of its parts, the initial rate of work of the forces on the system is zero.*

Conversely, if the forces on the system are such that, however the system is set in motion, the initial rate of work of the forces is zero, then the system is in equilibrium under the forces.

We shall prove the principle for a single rigid body. Choose any point O as origin of position vectors, and let the initial motion of the body be equivalent to a velocity $\mathbf{v}$ of the particle at O and an angular velocity $\mathbf{A}$ about it. Then the velocity of the particle at $\mathbf{r}$ is $\mathbf{v} + \mathbf{A}\times\mathbf{r}$. If $\mathbf{F}$ is the force acting at this particle, the initial rate of working of $\mathbf{F}$ is

$$\mathbf{F}\cdot\mathbf{v} + \mathbf{F}\cdot\mathbf{A}\times\mathbf{r}.$$

Considering all the forces on the body, we have for the total initial activity of the forces

$$\mathbf{v}\cdot\Sigma\mathbf{F} + \mathbf{A}\cdot\Sigma\mathbf{r}\times\mathbf{F}.$$

And, in virtue of the conditions of equilibrium of the body, this expression vanishes whatever the values of $\mathbf{v}$ and $\mathbf{A}$.

Conversely, if this activity is zero for all initial motions, the body must be in equilibrium. For, choosing one of translation only ($\mathbf{A} = 0$), we must have $\mathbf{v}\cdot\Sigma\mathbf{F}$ zero for all values of $\mathbf{v}$. Hence $\Sigma\mathbf{F} = 0$. Similarly choosing one of rotation only about O we

find $\mathbf{A}\cdot\Sigma\mathbf{r}\times\mathbf{F}$ zero for all values of $\mathbf{A}$. Therefore $\Sigma\mathbf{r}\times\mathbf{F}$ must also be zero, and the conditions of equilibrium are satisfied.

Such velocities as we have imagined given to the body are called *virtual velocities*; and the work done by the forces owing to these velocities is called *virtual work*. The above principle is frequently stated differently. Consider the infinitesimal *virtual displacement* of the body from the equilibrium position in the infinitesimal interval δt, owing to the virtual velocity imparted to it. The virtual work δW of the forces during this displacement is, to the first order,

$$\delta W = \frac{dW}{dt}\cdot\delta t.$$

And, if the body is in equilibrium, this vanishes not on account of the convergence of δt to the limit zero, but because the coefficient $\frac{dW}{dt}$ is zero, representing the initial activity. The value of δW is generally calculated, not in terms of δt, but of the small increments $\delta\theta$, $\delta\phi$, etc., of the coordinates expressing the position of the body.

Equilibrium of Strings and Wires.

105.* String under any forces. Let s be the length of the string measured from a fixed point A up to a variable point P, and δs the length of the element PP' (Fig 47). Suppose the string to be acted on by a force $\mathbf{F}$ per unit length, varying from point to point. Then $\mathbf{F}\,\delta s$ is the force on the element PP' due to external action. But if T and $T+\delta T$ are the values of the tensions at P and P' respectively, and $\mathbf{t}$ and $\mathbf{t}+\delta\mathbf{t}$ the unit tangents at those points, the element PP' is also acted on by the forces $-T\mathbf{t}$ and $(T+\delta T)(\mathbf{t}+\delta\mathbf{t})$. For equilibrium of the element the vector sum of these forces must be zero; that is

$$\delta T\,\mathbf{t} + T\,\delta\mathbf{t} + \delta T\,\delta\mathbf{t} + \mathbf{F}\,\delta s = 0.$$

Dividing throughout by δs, and taking limiting values as $\delta s \to 0$, we have

$$\frac{dT}{ds}\mathbf{t} + T\frac{d\mathbf{t}}{ds} + \mathbf{F} = 0,$$

which, by Art 59, is equivalent to

$$\frac{dT}{ds}\mathbf{t} + T\kappa\mathbf{n} + \mathbf{F} = 0, \quad\text{..............................}(1)$$

where κ is the curvature at P and $\mathbf{n}$ the unit normal at that point. The force $\mathbf{F}$ must therefore be in the plane of $\mathbf{t}$ and $\mathbf{n}$, *i.e.* the plane of curvature. If we write

$$\mathbf{F} = F_1\mathbf{t} + F_2\mathbf{n},$$

we may replace the single vector equation (1) by the two scalar equations

$$\left.\begin{aligned} \frac{dT}{ds} + F_1 = 0, \\ \kappa T + F_2 = 0. \end{aligned}\right\} \qquad \text{..........(2)}$$

These are the *equations of equilibrium* for the string.

106.* Wire or thin rod under any forces. In the case of a wire or rod the stress across any section is not as a rule tangential, nor can it be represented by a single force. In general it consists of a force and a couple due to bending and twisting. Let $\mathbf{S}$ and $\mathbf{L}$ represent the force and couple respectively at the point P, acting on the end of the portion AP. Then the element PP' (Fig. 47), of length δs, is acted on by forces $-\mathbf{S}$ at P and $\mathbf{S}+\delta\mathbf{S}$ at P', and by bending couples of torque $-\mathbf{L}$ and $\mathbf{L}+\delta\mathbf{L}$; and in addition there is a force $\mathbf{F}\,\delta s$ and a couple of torque $\mathbf{G}\,\delta s$ due to external action on the element.

For the equilibrium of the rigid element PP' the vector sum of the forces must be zero, and the torque about any point (say P) also zero. Hence the equations

$$\delta\mathbf{S} + \mathbf{F}\,\delta s = 0$$

and

$$\delta\mathbf{L} + \mathbf{G}\,.\,\delta s + \delta\mathbf{r}\times(\mathbf{S}+\delta\mathbf{S}) = 0,$$

where $\delta\mathbf{r}$ is the vector $\overrightarrow{PP'}$. Dividing throughout by δs, and proceeding to the limit, we have the *equations of equilibrium* in the form

$$\left\{\begin{aligned} &\frac{d\mathbf{S}}{ds} + \mathbf{F} = 0, \qquad \text{..........(1)} \\ &\frac{d\mathbf{L}}{ds} + \mathbf{t}\times\mathbf{S} + \mathbf{G} = 0. \qquad \text{..........(2)} \end{aligned}\right.$$

These may, if desired, be replaced by six Cartesian equations, got by resolving along suitable axes. The rectangular set $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$, varying from point to point, is most convenient. Thus, if we put

$$\mathbf{S} = S_1\mathbf{t} + S_2\mathbf{n} + S_3\mathbf{b},$$

and so on, and remember that

$$\frac{d\mathbf{t}}{ds} = \kappa\mathbf{n}, \quad \frac{d\mathbf{b}}{ds} = -\lambda\mathbf{n}, \quad \frac{d\mathbf{n}}{ds} = \lambda\mathbf{b} - \kappa\mathbf{t},$$

we find in place of (1) and (2) the six equations

$$\left.\begin{aligned} &\frac{dS_1}{ds} - \kappa S_2 + F_1 = 0, \\ &\frac{dS_2}{ds} + \kappa S_1 - \lambda S_3 + F_2 = 0, \\ &\frac{dS_3}{ds} + \lambda S_2 + F_3 = 0, \end{aligned}\right\} \quad \text{.........................(3)}$$

and

$$\left.\begin{aligned} &\frac{dL_1}{ds} - \kappa L_2 + G_1 = 0, \\ &\frac{dL_2}{ds} + \kappa L_1 - \lambda L_3 - S_3 + G_2 = 0, \\ &\frac{dL_3}{ds} + \lambda L_2 + S_2 + G_3 = 0. \end{aligned}\right\} \quad \text{..................(4)}$$

For a *plane system* $\lambda = 0 = S_3 = F_3$, and the axes of the couples L and G are in the directions of $\mathbf{b}$, so that L_1, L_2, G_1, G_2 all vanish. If then $L_3 = L$ and $G_3 = G$ the equations reduce to

$$\left.\begin{aligned} &\frac{dS_1}{ds} - \kappa S_2 + F_1 = 0, \\ &\frac{dS_2}{ds} + \kappa S_1 + F_2 = 0, \\ &\frac{dL}{ds} + S_2 + G = 0. \end{aligned}\right\} \quad \text{.............................(5)}$$

If in addition the system is *perfectly flexible*, the bending moment L is zero. This makes $S_2 + G = 0$; so that, if there is no external couple G, the normal resolute S_2 must vanish, making the stress purely tangential, as found in the previous Art.

EXERCISES ON CHAPTER VIII.

1. A body is in equilibrium under four forces acting along the sides of a cyclic quadrilateral. Prove that the forces are proportional to the lengths of the opposite sides.

2. If four forces acting along the sides of a cyclic quadrilateral are inversely proportional to the lengths of those sides, show that

their resultant acts along the line joining the intersections of opposite sides.

3. Forces act along the sides of a quadrilateral proportional to p, q, r, s times the lengths of those sides. Show that, if the body is in equilibrium, $pr=qs$, and the ratios $p:q$ and $q:r$ are the ratios in which the diagonals divide each other.

4. Four forces act along the sides of a skew quadrilateral represented by $a\,.\,\overrightarrow{AB}$, $b\,.\,\overrightarrow{BC}$, $c\,.\,\overrightarrow{CD}$, $d\,.\,\overrightarrow{DA}$ respectively. Show that they cannot be in equilibrium. If $a=b=c=d$ they are equivalent to a couple whose plane is parallel to the diagonals AC, BD. But if $ac=bd$ they have a resultant whose line of action intersects the diagonals.

5. Forces act along the sides of a skew polygon taken in order, proportional to the lengths of the sides along which they act. Show that they are equivalent to a couple.

6. Show that a body cannot be in equilibrium under six forces acting along the edges of a tetrahedron

7. Six equal forces act along the edges of a regular tetrahedron $ABCD$ in the directions AB, BC, CA, DA, DB, DC. Prove that their central axis is the perpendicular from D to the face ABC.

8. If in the previous exercise the tetrahedron is not regular, and the forces are proportional to the lengths of the edges along which they act, their central axis is parallel to the line joining D to the centroid G of the face ABC. and if ϕ is the inclination of these parallel lines to that face, their distance apart is $\frac{4}{3}\Delta\cos\phi/DG$, Δ being the area of the triangle ABC.

9. Forces act at the vertices of a tetrahedron outward, perpendicular to the opposite faces and proportional to their areas. Prove that the body on which they act is in equilibrium.

10. Twelve equal forces act along the edges of a cube, the parallel forces having the same sense. Prove that their central axis is a diagonal. If, instead of forces, there are twelve equal couples whose planes are parallel to the faces of the cube, show that their central axis is parallel to a diagonal.

11. Show that, for any system of forces, the couple $\mathbf{G}$ is least for points on the central axis.

12. Forces through the points $A_1, A_2, \ldots, A_n$ are represented by the vectors $\overrightarrow{A_1A_1'}$, $\overrightarrow{A_2A_2'}, \ldots, \overrightarrow{A_nA_n'}$ respectively. If G is the centroid of the n points A_m, and G' that of the n points A_m', show that the central axis of the forces is parallel to GG'. If the lines of action of the forces intersect any plane perpendicular to the central

axis in $B_1, B_2, \ldots, B_n$, show that the central axis meets this plane in the centroid of these points with associated numbers proportional to the resolutes of the forces in the direction of the central axis.

13. Forces act at the middle points of the sides of a plane polygon, in the plane of the figure and perpendicularly to the sides. If they are proportional to the lengths of the sides, and act either all inward or all outward, prove that the body is in equilibrium.

14. Forces act at the centroids of the faces of a closed polyhedron, proportional to the areas of the faces. If they are normal to the faces, and either all inward or all outward, prove that the system is in equilibrium.

15. Extend the previous exercise to the case of a closed curved surface, by increasing the number of faces indefinitely.

16. If four forces are in equilibrium, show that the invariant Γ of any two is equal to that of the other two. Also that the same invariant of any three is zero.

17. If the origin is taken on the axis of the equivalent wrench (pitch p), and $\mathbf{k}$ is the unit vector in the direction of the axis, show that the null plane at the point $p(B\mathbf{i}+A\mathbf{j})+C\mathbf{k}$ is

$$\mathbf{r}\cdot(A\mathbf{i}-B\mathbf{j}-\mathbf{k})+C=0.$$

Hence prove that, if a plane meets the central axis in P and makes an angle ϕ with it, its null point Q is such that PQ is perpendicular to the axis and of length $p \cot \phi$.

18. If two straight lines intersect in a point P, their conjugates also intersect and lie in the null plane at P.

19. Show that the null planes of collinear points have a common line of intersection.

20. Prove that any system of forces acting on a rigid body can be replaced by two equal forces equally inclined to the central axis.

21. A transversal intersects the lines of action of two conjugate forces. Prove that either point of intersection is the null point of the plane containing the transversal and the other line of action.

22. Any two conjugate lines intersect a plane in P and Q. Show that PQ passes through the null point of the plane.

23. A system of forces is reduced to three, acting at fixed points A, B, C. If the force at A is fixed in direction, prove that each of the other two lies in a fixed plane. Also that these planes intersect in the line BC.

24. A rigid body is acted on by a force $\mathbf{F}$ per unit mass, varying from point to point. If, relative to an origin O, $\mathbf{r}$ is the position vector

of the point P where the density is μ, show that the whole action is equivalent to a force $\int \mu \mathbf{F}\, dv$ acting at O, together with a couple whose torque is $\int \mu \mathbf{r}\times\mathbf{F}\, dv$, where dv is the element of volume at P.

25. Deduce from Art. 105 the Cartesian equations of equilibrium of a string,
$$\frac{d}{ds}\left[T\frac{dx}{ds}\right]+X=0,$$
etc., where X, Y, Z are the resolutes of the force per unit length acting on the string.

26. A string is in equilibrium in the form of a helix, and the tension is constant throughout the string. Prove that the force on any element is directly from the axis of the helix.

27. A heavy string is suspended from two points, and hangs partly immersed in a fluid. Show that the curvatures of the portions just inside and just outside the fluid are as $D-D' : D$, where D, D' are the densities of the string and fluid respectively.

28. A heavy string is suspended from two fixed points, and the density is such that the form of the string is an equiangular spiral. Show that the density at any point P is inversely proportional to $r\cos^2\psi$, where r is the distance of P from the pole and ψ the angle the tangent at P makes with the horizontal

29. A heavy uniform string rests on a smooth curve in a vertical plane, and is acted on by forces at its ends. Prove that the difference between the tensions at any two points is equal to the weight of a string whose length is the vertical distance between the points. Also find the pressure on the curve at any point.

If the string is light, show that the tension is constant, and that the pressure varies as the curvature.

30. A light string rests on a rough curve in a state bordering on motion. Show that the ratio of the tensions at any two points is $e^{\mu\theta}$, where μ is the coefficient of friction and θ the angle between the tangents at the two points.

31. A heavy string, resting on a rough curve in a vertical plane, is on the point of motion. Write down equations for determining the tension and pressure at any point.

32. A rigid body is subjected to fluid pressure of intensity p, variable over the surface. Show that the total action of the fluid on the body is equivalent to a force $-\int p\mathbf{n}\, dS$ through the origin, together with a torque $-\int p\mathbf{r}\times\mathbf{n}\, dS$, where $\mathbf{n}$ is the unit outward normal and dS the area of an element of the surface.

SUMMARY.

ADDITION AND SUBTRACTION.

Vectors are compounded according to the triangle **law of addition.** Thus, if three points O, P, R are chosen so that $\overrightarrow{OP} = \mathbf{a}$ and $\overrightarrow{PR} = \mathbf{b}$, the vector $\overrightarrow{OR}$ is the sum or resultant of $\mathbf{a}$ and $\mathbf{b}$. When several vectors are added together the commutative and associative laws hold. The individual vectors are called the *components* of the resultant.

The *negative* of $\mathbf{a}$ is the vector which has the same length as $\mathbf{a}$ but the opposite direction. It is denoted by $-\mathbf{a}$.

To **subtract** the vector $\mathbf{b}$ from $\mathbf{a}$ reverse the direction of $\mathbf{b}$ and add. Thus

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}).$$

If m is any positive real number, $m\mathbf{a}$ is defined to mean the vector in the same direction as $\mathbf{a}$, but of m times its length. Thus

$$m\mathbf{a} = m(a\hat{\mathbf{a}}) = (ma)\hat{\mathbf{a}}.$$

where $\hat{\mathbf{a}}$ is a *unit vector* and a the *module* of $\mathbf{a}$.

Similarly the vector $(-m)\mathbf{a}$ is defined to be the vector obtained by reversing the direction of $\mathbf{a}$ and multiplying its length by m.

The general laws of association and distribution for scalar multipliers hold as in ordinary algebra. Thus

$$m(n\mathbf{a}) = (mn)\mathbf{a} = n(m\mathbf{a}),$$
$$(m+n)\mathbf{a} = m\mathbf{a} + n\mathbf{a},$$
$$m(\mathbf{a} + \mathbf{b}) = m\mathbf{a} + m\mathbf{b}.$$

Any vector $\mathbf{a}$ can be expressed as the sum of three others, parallel to any three non-coplanar vectors. When these three components are mutually perpendicular they are called the

resolutes or *resolved parts* of $\mathbf{a}$ in those directions. We use the notation

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k},$$

where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are the unit vectors in those directions. Vectors may be compounded by adding their like components. Thus

$$\Sigma\mathbf{a} = (\Sigma a_1)\mathbf{i} + (\Sigma a_2)\mathbf{j} + (\Sigma a_3)\mathbf{k}.$$

The unit vector

$$\hat{\mathbf{a}} = \frac{1}{a}\mathbf{a} = (\cos\alpha)\mathbf{i} + (\cos\beta)\mathbf{j} + (\cos\gamma)\mathbf{k},$$

where $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosines of the vector.

The rectangular *coordinates* x, y, z of a point are connected with its position vector $\mathbf{r}$ by the relation

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

We speak of this point briefly as the point $\mathbf{r}$.

The line joining the points $\mathbf{a}$ and $\mathbf{b}$ is divided in the ratio $m : n$ by the point

$$\mathbf{r} = \frac{n\mathbf{a} + m\mathbf{b}}{m + n}.$$

The **centroid** of the points $\mathbf{a}_1$, $\mathbf{a}_2, \ldots$, with associated real numbers p_1, $p_2, \ldots$, is the point

$$\bar{\mathbf{r}} = \frac{\Sigma p\mathbf{a}}{\Sigma p}.$$

The **centre of mass** (c.m.) of particles m_1, $m_2, \ldots$ at the points $\mathbf{r}_1$, $\mathbf{r}_2, \ldots$ is the point

$$\bar{\mathbf{r}} = \frac{\Sigma m\mathbf{r}}{\Sigma m}.$$

This point coincides with the centre of gravity of the particles.

The vector **equation of the straight line** through the point $\mathbf{a}$ parallel to $\mathbf{b}$ is

$$\mathbf{r} = \mathbf{a} + t\mathbf{b}.$$

The straight line passing through the points $\mathbf{a}$ and $\mathbf{b}$ is

$$\mathbf{r} = (1 - t)\mathbf{a} + t\mathbf{b}.$$

The necessary and sufficient condition that three points should be *collinear* is that there exists a linear relation between their position vectors, in which the algebraic sum of the coefficients is equal to zero.

The bisectors of the angles between the straight lines $\mathbf{r} = t\mathbf{a}$ and $\mathbf{r} = t\mathbf{b}$ are

$$\mathbf{r} = t(\hat{\mathbf{a}} + \hat{\mathbf{b}}) = t\left(\frac{\mathbf{a}}{a} + \frac{\mathbf{b}}{b}\right)$$

and

$$\mathbf{r} = t(\hat{\mathbf{a}} - \hat{\mathbf{b}}) = t\left(\frac{\mathbf{a}}{a} - \frac{\mathbf{b}}{b}\right).$$

The plane through the point $\mathbf{a}$ parallel to $\mathbf{b}$ and $\mathbf{c}$ is

$$\mathbf{r} = \mathbf{a} + s\mathbf{b} + t\mathbf{c}.$$

The plane through the three points $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ is

$$\mathbf{r} = (1 - s - t)\mathbf{a} + s\mathbf{b} + t\mathbf{c}.$$

The necessary and sufficient condition that four points should be *coplanar* is that, in the linear relation between their position vectors, the algebraic sum of the coefficients be equal to zero.

The necessary and sufficient condition that a linear relation, connecting the position vectors of any number of fixed points, should be *independent of the origin*, is that the algebraic sum of the coefficients be zero.

The **vector area** of a plane figure is specified by a vector normal to the plane, with module equal to the measure of the area of the figure. *The sum of the vector areas of the faces of a closed polyhedron is zero.*

PRODUCTS OF VECTORS.

The **scalar product** of two vectors $\mathbf{a}$ and $\mathbf{b}$, whose directions are inclined at an angle θ, is the real number $ab \cos \theta$, and is written

$$\mathbf{a} \cdot \mathbf{b} = ab \cos \theta = \mathbf{b} \cdot \mathbf{a}.$$

The *condition of perpendicularity* of $\mathbf{a}$ and $\mathbf{b}$ is

$$\mathbf{a} \cdot \mathbf{b} = 0.$$

The *square* of the vector $\mathbf{a}$ is

$$\mathbf{a}^2 = \mathbf{a} \cdot \mathbf{a} = a^2.$$

Also, with the usual notation,

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3,$$

$$\mathbf{a}^2 = a_1{}^2 + a_2{}^2 + a_3{}^2,$$

and

$$\hat{\mathbf{a}} \cdot \hat{\mathbf{b}} = \cos \theta = l_1 l_2 + m_1 m_2 + n_1 n_2,$$

where l_1, m_1, n_1 and l_2, m_2, n_2 are the direction cosines of $\mathbf{a}$ and $\mathbf{b}$ respectively.

For the mutually perpendicular unit vectors **i**, **j**, **k**,

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = 1,$$
$$\mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = 0.$$

Any vector **r** may be expressed as the sum of two vectors

$$\frac{\mathbf{a}\cdot\mathbf{r}}{a^2}\mathbf{a} \quad \text{and} \quad \left(\mathbf{r} - \frac{\mathbf{a}\cdot\mathbf{r}}{a^2}\mathbf{a}\right)$$

respectively, parallel and perpendicular to **a**. Also

$$\mathbf{r} = \mathbf{r}\cdot\mathbf{i}\mathbf{i} + \mathbf{r}\cdot\mathbf{j}\mathbf{j} + \mathbf{r}\cdot\mathbf{k}\mathbf{k}.$$

The *distributive law* holds for scalar products

$$\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c}.$$

The **vector product** of two vectors **a** and **b**, whose directions are inclined at an angle θ, is the vector whose module is $ab \sin\theta$, and whose direction is perpendicular to both **a** and **b**, being positive relative to a rotation from **a** to **b**. We write it

$$\mathbf{a}\times\mathbf{b} = ab\sin\theta\,\hat{\mathbf{n}} = -\mathbf{b}\times\mathbf{a}.$$

Its value may also be expressed

$$\mathbf{a}\times\mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$
$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ \mathbf{i} & \mathbf{j} & \mathbf{k} \end{vmatrix}.$$

The *condition of parallelism* of **a** and **b** is

$$\mathbf{a}\times\mathbf{b} = 0.$$

For the unit vectors **i**, **j**, **k** we have

$$\mathbf{i}\times\mathbf{i} = \mathbf{j}\times\mathbf{j} = \mathbf{k}\times\mathbf{k} = 0,$$

while

$$\mathbf{i}\times\mathbf{j} = \mathbf{k} = -\mathbf{j}\times\mathbf{i},$$
$$\mathbf{j}\times\mathbf{k} = \mathbf{i} = -\mathbf{k}\times\mathbf{j},$$
$$\mathbf{k}\times\mathbf{i} = \mathbf{j} = -\mathbf{i}\times\mathbf{k}.$$

The *distributive law* holds for vector products also; but the order of the factors in each term must be maintained.

The **scalar triple product** of three vectors **a**, **b**, **c** is the scalar product of **a** and **b**×**c**. It is the measure of the volume of the parallelepiped whose edges are determined by the three vectors. The value of the product is unaltered by interchanging the dot

and the cross, or by changing the order of the factors, provided the cyclic order is unaltered. Thus

$$\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}=\mathbf{a}\times\mathbf{b}\cdot\mathbf{c}=\mathbf{c}\cdot\mathbf{b}\times\mathbf{a},$$

and so on. The product is generally written

$$[\mathbf{abc}],$$

a notation which indicates the three vectors and the cyclic order. If, however, the cyclic order of the factors is changed, the sign of the product is changed. Thus

$$[\mathbf{abc}]=-[\mathbf{acb}].$$

The value of the product is given by the determinant

$$[\mathbf{abc}]=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

The **vector triple product** $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ is the vector product of $\mathbf{a}$ and $\mathbf{b}\times\mathbf{c}$. It is a vector in the plane of $\mathbf{b}$ and $\mathbf{c}$, and its value is given by

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=\mathbf{a}\cdot\mathbf{c}\,\mathbf{b}-\mathbf{a}\cdot\mathbf{b}\,\mathbf{c}.$$

The position of the brackets in this product is not arbitrary; for $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$ is a vector in the plane of $\mathbf{a}$ and $\mathbf{b}$, and its value is

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c}=\mathbf{a}\cdot\mathbf{c}\,\mathbf{b}-\mathbf{b}\cdot\mathbf{c}\,\mathbf{a}.$$

Neither can the order of the factors be changed at pleasure.

The **scalar quadruple product** $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})$ is the scalar product of $\mathbf{a}\times\mathbf{b}$ and $\mathbf{c}\times\mathbf{d}$. Its value is given by

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})=\mathbf{a}\cdot\mathbf{c}\,\mathbf{b}\cdot\mathbf{d}-\mathbf{a}\cdot\mathbf{d}\,\mathbf{b}\cdot\mathbf{c}$$
$$=\begin{vmatrix} \mathbf{a}\cdot\mathbf{c} & \mathbf{a}\cdot\mathbf{d} \\ \mathbf{b}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{d} \end{vmatrix}.$$

The **vector quadruple product** $(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})$ may be expanded in terms either of $\mathbf{a}$ and $\mathbf{b}$ or of $\mathbf{c}$ and $\mathbf{d}$. Thus

$$(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})=[\mathbf{acd}]\mathbf{b}-[\mathbf{bcd}]\mathbf{a}$$
$$=[\mathbf{abd}]\mathbf{c}-[\mathbf{abc}]\mathbf{d}.$$

Equating these two expressions for the product, and writing $\mathbf{r}$ instead of $\mathbf{d}$, we see that any vector $\mathbf{r}$ is expressible in terms of three non-coplanar vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ by the formula

$$\mathbf{r}=\frac{[\mathbf{rbc}]\mathbf{a}+[\mathbf{rca}]\mathbf{b}+[\mathbf{rab}]\mathbf{c}}{[\mathbf{abc}]}.$$

The **reciprocal system** of vectors to the non-coplanar system $\mathbf{a}, \mathbf{b}, \mathbf{c}$ is

$$\mathbf{a}' = \frac{\mathbf{b}\times\mathbf{c}}{[\mathbf{abc}]}; \quad \mathbf{b}' = \frac{\mathbf{c}\times\mathbf{a}}{[\mathbf{abc}]}; \quad \mathbf{c}' = \frac{\mathbf{a}\times\mathbf{b}}{[\mathbf{abc}]},$$

which satisfy the relations

$$\mathbf{a}\cdot\mathbf{a}' = \mathbf{b}\cdot\mathbf{b}' = \mathbf{c}\cdot\mathbf{c}' = 1$$

and

$$\mathbf{a}\cdot\mathbf{b}' = \mathbf{a}\cdot\mathbf{c}' = \text{etc.} = 0.$$

The reciprocal system to $\mathbf{a}', \mathbf{b}', \mathbf{c}'$ is $\mathbf{a}, \mathbf{b}, \mathbf{c}$. The above expression for $\mathbf{r}$ in terms of $\mathbf{a}, \mathbf{b}, \mathbf{c}$ may be written

$$\mathbf{r} = \mathbf{r}\cdot\mathbf{a}'\mathbf{a} + \mathbf{r}\cdot\mathbf{b}'\mathbf{b} + \mathbf{r}\cdot\mathbf{c}'\mathbf{c}.$$

The system $\mathbf{i}, \mathbf{j}, \mathbf{k}$ is its own reciprocal.

THE PLANE AND THE STRAIGHT LINE.

The standard form of the **equation of a plane** perpendicular to $\mathbf{n}$ is

$$\mathbf{r}\cdot\mathbf{n} = q.$$

The *perpendicular distance* from the point $\mathbf{r}'$ to this plane is

$$p = \frac{q - \mathbf{r}'\cdot\mathbf{n}}{n}.$$

The distance measured parallel to the vector $\hat{\mathbf{b}}$ is

$$l = \frac{q - \mathbf{r}'\cdot\mathbf{n}}{\mathbf{n}\cdot\hat{\mathbf{b}}}$$

The plane through the point $\mathbf{d}$ perpendicular to $\mathbf{n}$ is

$$\mathbf{r}\cdot\mathbf{n} = \mathbf{d}\cdot\mathbf{n}.$$

The planes bisecting the angles between the two planes

$$\left.\begin{aligned}\mathbf{r}\cdot\mathbf{n} &= q,\\ \mathbf{r}\cdot\mathbf{n}' &= q'\end{aligned}\right\} \quad \text{...(A)}$$

are

$$\mathbf{r}\cdot\left[\frac{\mathbf{n}}{n} \mp \frac{\mathbf{n}'}{n'}\right] = \frac{q}{n} \mp \frac{q'}{n'}.$$

The equation of *any plane through the line of intersection* of the planes (A) is expressible as

$$\mathbf{r}\cdot(\mathbf{n} - \lambda\mathbf{n}') = q - \lambda q'.$$

This plane may be made to satisfy one other condition by giving a suitable real value to the parameter λ.

The plane containing the three points $\mathbf{a}, \mathbf{b}, \mathbf{c}$ is

$$\mathbf{r}\cdot(\mathbf{b}\times\mathbf{c} + \mathbf{c}\times\mathbf{a} + \mathbf{a}\times\mathbf{b}) = [\mathbf{abc}].$$

The plane through the point $\mathbf{a}$ parallel to $\mathbf{b}$ and $\mathbf{c}$ is

$$\mathbf{r}\cdot\mathbf{b}\times\mathbf{c}=[\mathbf{abc}]$$

or

$$[\mathbf{r}-\mathbf{a},\ \mathbf{b},\ \mathbf{c}]=0.$$

The plane through the points $\mathbf{a}$ and $\mathbf{b}$, and parallel to $\mathbf{c}$, is

$$\mathbf{r}\cdot(\mathbf{b}-\mathbf{a})\times\mathbf{c}=[\mathbf{abc}].$$

The plane containing the straight line $\mathbf{r}=\mathbf{a}+t\mathbf{b}$ and the point $\mathbf{c}$ is

$$\mathbf{r}\cdot(\mathbf{a}-\mathbf{c})\times\mathbf{b}=[\mathbf{abc}].$$

The *perpendicular* from the point $\mathbf{r}'$ to the straight line

$$\mathbf{r}=\mathbf{a}+t\mathbf{b}$$

is

$$\mathbf{p}=\mathbf{a}-\mathbf{r}'-\frac{1}{b^2}\mathbf{b}\cdot(\mathbf{a}-\mathbf{r}')\mathbf{b}.$$

The **condition of intersection** of the straight lines

$$\left.\begin{aligned}\mathbf{r}&=\mathbf{a}+t\mathbf{b},\\ \mathbf{r}&=\mathbf{a}'+s\mathbf{b}'\end{aligned}\right\}\quad\text{..................................(B)}$$

is

$$[\mathbf{b},\ \mathbf{b}',\ \mathbf{a}-\mathbf{a}']=0.$$

The length of the **common perpendicular** to the two lines is

$$p=\frac{\mathbf{n}\cdot(\mathbf{a}-\mathbf{a}')}{n}=\frac{1}{n}[\mathbf{b},\ \mathbf{b}',\ \mathbf{a}-\mathbf{a}'],$$

where $\mathbf{n}=\mathbf{b}\times\mathbf{b}'$ and $n=\text{mod. }\mathbf{n}$. The common perpendicular is the line of intersection of the planes

$$[\mathbf{r}-\mathbf{a},\ \mathbf{b},\ \mathbf{b}\times\mathbf{b}']=0,$$
$$[\mathbf{r}-\mathbf{a}',\ \mathbf{b}',\ \mathbf{b}\times\mathbf{b}']=0.$$

Plucker's coordinates of a line are the *unit* vector $\mathbf{d}$ parallel to the line, and the moment $\mathbf{m}$ about the origin of this vector localised in the line.

The *mutual moment* of the two straight lines $\mathbf{d}$, $\mathbf{m}$ and $\mathbf{d}'$, $\mathbf{m}'$ is

$$M=\mathbf{m}\cdot\mathbf{d}'+\mathbf{m}'\cdot\mathbf{d}.$$

This is connected with the length p of their common perpendicular by the equation $\quad M=p\sin\theta,$

θ being the angle of inclination of the two lines. The lines intersect if $M=0$.

If the position vectors of three vertices of a tetrahedron relative to the other vertex are $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, the **volume of the tetrahedron** is

$$V=\tfrac{1}{6}[\mathbf{abc}].$$

THE SPHERE.

The **equation of the sphere** of radius a with centre at the point $\mathbf{c}$ is

$$(\mathbf{r}-\mathbf{c})^2 = a^2$$

or

$$\mathbf{r}^2 - 2\mathbf{r}\cdot\mathbf{c} + k = 0,$$

where $k = \mathbf{c}^2 - a^2$.

The equation of the **tangent plane** at the point $\mathbf{d}$ is

$$\mathbf{r}\cdot\mathbf{d} - \mathbf{c}\cdot(\mathbf{r}+\mathbf{d}) + k = 0.$$

The condition that the plane $\mathbf{r}\cdot\mathbf{n} = q$ should touch the sphere is

$$(q - \mathbf{c}\cdot\mathbf{n})^2 = n^2(\mathbf{c}^2 - k).$$

The condition that the two spheres

$$\left.\begin{aligned} F(\mathbf{r}) &\equiv \mathbf{r}^2 - 2\mathbf{r}\cdot\mathbf{c} + k = 0, \\ F'(\mathbf{r}) &\equiv \mathbf{r}^2 - 2\mathbf{r}\cdot\mathbf{c}' + k' = 0 \end{aligned}\right\} \quad \text{............................(C)}$$

should cut each other orthogonally is

$$2\mathbf{c}\cdot\mathbf{c}' = k + k'.$$

The **polar plane** of the point $\mathbf{h}$ with respect to the first sphere is

$$\mathbf{r}\cdot\mathbf{h} - \mathbf{c}\cdot(\mathbf{r}+\mathbf{h}) + k = 0.$$

Any straight line drawn through the point $\mathbf{h}$ to intersect the sphere is cut harmonically by the surface and the polar plane of $\mathbf{h}$.

If the polar plane of the point $\mathbf{h}$ passes through the point $\mathbf{g}$, then the polar plane of $\mathbf{g}$ passes through $\mathbf{h}$.

The **radical plane** of the two spheres (C) is

$$F(\mathbf{r}) = F'(\mathbf{r}),$$

that is

$$2\mathbf{r}\cdot(\mathbf{c}-\mathbf{c}') = k - k'.$$

The tangents to the two spheres from any point on this plane are equal in length. Also

$$F(\mathbf{r}) - \lambda F'(\mathbf{r}) = 0$$

represents a *system of spheres* with a common radical plane perpendicular to the line of centres.

DIFFERENTIATION AND INTEGRATION

If $\mathbf{r}$ is a function of a scalar variable t, and $\delta\mathbf{r}$ is the increment in $\mathbf{r}$ corresponding to the increment δt in t, then the limiting value of the quotient $\delta\mathbf{r}/\delta t$ as δt tends to zero is called the *derivative* of $\mathbf{r}$ with respect to t. We use the notation

$$\operatorname*{Lt}_{\delta t\to 0}\frac{\delta\mathbf{r}}{\delta t}=\frac{d\mathbf{r}}{dt}.$$

The derivative of this function is called the second derivative, and so on.

The rules for differentiating **sums and products** of vectors are similar to those for algebraic sums and products. Thus

$$\frac{d}{dt}(\mathbf{r}+\mathbf{s}+\ldots)=\frac{d\mathbf{r}}{dt}+\frac{d\mathbf{s}}{dt}+\ldots,$$

$$\frac{d}{dt}(\mathbf{r}\cdot\mathbf{s})=\frac{d\mathbf{r}}{dt}\cdot\mathbf{s}+\mathbf{r}\cdot\frac{d\mathbf{s}}{dt},$$

$$\frac{d}{dt}(\mathbf{r}\times\mathbf{s})=\frac{d\mathbf{r}}{dt}\times\mathbf{s}+\mathbf{r}\times\frac{d\mathbf{s}}{dt}.$$

Differentiating both sides of the equality $\mathbf{r}^2=r^2$, and using the second of these formulae, we obtain

$$\mathbf{r}\cdot\frac{d\mathbf{r}}{dt}=r\frac{dr}{dt},$$

which is a useful result. In particular, if $\mathbf{a}$ is a vector of constant length,

$$\mathbf{a}\cdot\frac{d\mathbf{a}}{dt}=0,$$

showing that $\mathbf{a}$ is perpendicular to its derivative. It is also worth noticing that

$$\frac{d}{dt}\left[\mathbf{r}\times\frac{d\mathbf{r}}{dt}\right]=\mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2}$$

and

$$\frac{d\mathbf{r}}{dt}=\frac{dx}{dt}\mathbf{i}+\frac{dy}{dt}\mathbf{j}+\frac{dz}{dt}\mathbf{k}.$$

To differentiate a triple product, or one involving several factors, differentiate each factor in turn. Thus

$$\frac{d}{dt}[\mathbf{abc}]=\left[\frac{d\mathbf{a}}{dt}\mathbf{bc}\right]+\left[\mathbf{a}\frac{d\mathbf{b}}{dt}\mathbf{c}\right]+\left[\mathbf{ab}\frac{d\mathbf{c}}{dt}\right],$$

and so on.

Integration is the reverse process to differentiation. The vector $\mathbf{F}$, whose derivative with respect to t is equal to $\mathbf{r}$, is called the *integral* of $\mathbf{r}$, and is written

$$\mathbf{F} = \int \mathbf{r}\, dt.$$

A constant of integration may be introduced as in algebraic calculus. Thus

$$2\int \mathbf{r}\cdot\frac{d\mathbf{r}}{dt}\, dt = \mathbf{r}^2 + c = r^2 + c,$$

$$\int \mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2}\, dt = \mathbf{r}\times\frac{d\mathbf{r}}{dt} + \mathbf{c}.$$

The equation $\dfrac{d^2\mathbf{r}}{dt^2} = -n^2\mathbf{r}$ may be integrated after scalar multiplication of both members with $2\dfrac{d\mathbf{r}}{dt}$ We then obtain

$$\left(\frac{d\mathbf{r}}{dt}\right)^2 = c - n^2\mathbf{r}^2.$$

A **definite integral** is defined as the limit of a sum, as in ordinary calculus. Cf. Arts. 61, 62.

GEOMETRY OF CURVES.

If $\mathbf{r}$ is the position vector of a current point on the curve, and s the length of the arc up to that point,

$$\mathbf{t} = \frac{d\mathbf{r}}{ds} = \frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j} + \frac{dz}{ds}\mathbf{k}$$

is the unit vector parallel to the tangent, called briefly the **unit tangent**. Hence the *equation of the tangent* is

$$\mathbf{R} = \mathbf{r} + u\mathbf{t}.$$

Further,
$$\frac{d^2\mathbf{r}}{ds^2} = \frac{d\mathbf{t}}{ds} = \kappa\mathbf{n},$$

where $\mathbf{n}$ is the unit vector parallel to the principal normal, and κ the *curvature* or arc-rate of turning of the tangent. We call $\mathbf{n}$ briefly the **unit normal**. From the last equation it follows that

$$\kappa^2 = \left(\frac{d^2\mathbf{r}}{ds^2}\right)^2 = \left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2.$$

The equation of the *principal normal* is

$$\mathbf{R} = \mathbf{r} + u\mathbf{n}.$$

The *normal plane* is the plane through the point $\mathbf{r}$ perpendicular to the tangent. Its equation is

$$(\mathbf{R}-\mathbf{r})\cdot\mathbf{t}=0.$$

The *osculating plane*, or *plane of curvature*, is that which contains the tangent and the principal normal. Its equation is

$$[\mathbf{R}-\mathbf{r},\ \mathbf{t},\ \mathbf{n}]=0.$$

The *binormal* is the straight line through the point $\mathbf{r}$ perpendicular to the plane of curvature. The **unit binormal** is

$$\mathbf{b}=\mathbf{t}\times\mathbf{n},$$

which has a derivative $$\frac{d\mathbf{b}}{ds}=-\lambda\mathbf{n},$$

where λ is the *torsion*, or arc-rate of turning of the binormal. Its value is given by

$$\lambda=\frac{1}{\kappa^2}\left[\frac{d\mathbf{r}}{ds},\ \frac{d^2\mathbf{r}}{ds^2},\ \frac{d^3\mathbf{r}}{ds^3}\right].$$

The derivative of the unit normal is

$$\frac{d\mathbf{n}}{ds}=-\kappa\mathbf{t}+\lambda\mathbf{b},$$

and the equation of the binormal is

$$\mathbf{R}=\mathbf{r}+u\frac{d\mathbf{r}}{ds}\times\frac{d^2\mathbf{r}}{ds^2}.$$

PARTICLE KINEMATICS AND DYNAMICS.

The **velocity** of a particle, whose position vector is $\mathbf{r}$, is

$$\mathbf{v}=\frac{d\mathbf{r}}{dt},$$

and its **acceleration** is the rate of increase of its velocity; or

$$\mathbf{a}=\frac{d\mathbf{v}}{dt}=\frac{d^2\mathbf{r}}{dt^2}.$$

Velocities are compounded by vector addition; and the same is true of accelerations.

The velocity of a particle *moving in a curve* with speed v is

$$\mathbf{v}=v\mathbf{t},$$

and its acceleration is $$\mathbf{a}=\frac{dv}{dt}\mathbf{t}+\kappa v^2\mathbf{n}.$$

For a particle moving in a plane curve the radial and transverse resolutes of the velocity are

$$\frac{dr}{dt} \quad \text{and} \quad r\frac{d\theta}{dt}$$

respectively ; and those of the acceleration are

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 \quad \text{and} \quad r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}.$$

The **areal velocity** of a particle about the origin is

$$\tfrac{1}{2}\mathbf{r}\times\mathbf{v} = \tfrac{1}{2}h\mathbf{k},$$

where $\mathbf{k}$ is the unit vector perpendicular to $\mathbf{r}$ and $\mathbf{v}$, and

$$h = pv = r^2\frac{d\theta}{dt},$$

p being the perpendicular distance from the origin to the tangent to the path of the particle.

The linear **momentum** of a moving particle of mass m is

$$\mathbf{M} = m\mathbf{v},$$

and its rate of increase is

$$\frac{d\mathbf{M}}{dt} = m\frac{d\mathbf{v}}{dt} = m\mathbf{a}.$$

Newton's **second law of motion** states that the force $\mathbf{F}$ acting on a particle has the direction of, and is proportional to, the rate of increase of the particle's momentum. Hence, with the appropriate unit of force,

$$\mathbf{F} = \frac{d}{dt}(m\mathbf{v}) = m\mathbf{a},$$

the mass of the particle being assumed constant.

The **impulse** of a force $\mathbf{F}$ acting during the interval t_0 to t_1 is

$$\mathbf{I} = \int_{t_0}^{t_1}\mathbf{F}\,dt = m(\mathbf{v}_1 - \mathbf{v}_0),$$

and is therefore equal to the increase of momentum produced by it.

The **activity** of the force is its rate of working. Its value $\mathbf{F}\cdot\mathbf{v}$ is equal to the rate of increase of the *kinetic energy* of the particle. Thus

$$\mathbf{F}\cdot\mathbf{v} = \frac{dT}{dt} = \frac{d}{dt}(\tfrac{1}{2}mv^2).$$

The **moment or torque of a force F** about the origin is $\mathbf{r}\times\mathbf{F}$, where $\mathbf{r}$ is any point on its line of action. The **moment of momentum** of a particle about the origin is similarly

$$\mathbf{H} = \mathbf{r}\times(m\mathbf{v}),$$

$\mathbf{r}$ being the position vector of the particle. The term *angular momentum* (A.M.) is synonymous with moment of momentum. The moment of the force acting on a particle is equal to the rate of increase of the A.M.; or

$$\frac{d\mathbf{H}}{dt}=\mathbf{r}\times\mathbf{F}.$$

A **central force** is one acting always towards a fixed point called the *centre of force*. The orbit of a particle acted on by a central force is a plane orbit, the plane of the orbit containing the centre of force. The A.M. of the particle about the centre remains constant.

In the case of a central force $-\frac{\mu m}{r^2}\hat{\mathbf{r}}$, varying *inversely as the square of the distance*, the orbit is a conic with focus at the centre of force. If V is the speed at a point distant c from the centre of force, the eccentricity of the orbit is given by

$$e^2=\frac{h^2}{\mu}\left(\frac{V^2}{\mu}-\frac{2}{c}\right)+1.$$

If $V^2<2\mu/c$ the orbit is an *ellipse*, as in the case of the planets. If a, b are the semi-axes of the ellipse, the periodic time is

$$T=\frac{2\pi}{\sqrt{\mu}}a^{\frac{3}{2}},$$

and the speed at any point is given by

$$v^2=\mu\left(\frac{2}{r}-\frac{1}{a}\right),$$

while $$h^2=\frac{\mu b^2}{a}.$$

In the case of a central force $-\mu m\mathbf{r}$, varying *directly as the distance*, the orbit is an ellipse with centre at the centre of force O. The periodic time is now

$$T=\frac{2\pi}{\sqrt{\mu}},$$

and the speed at any point P is given by

$$v^2=\mu(a^2+b^2-r^2)=\mu\,.\,OD^2,$$

where OD is the semi-diameter conjugate to OP. Also

$$h=\sqrt{\mu}\,.\,ab.$$

If a particle acted on by a force $\mathbf{F}$ is **constrained to move on a smooth curve,** the equation of motion is

$$m\mathbf{a} = \mathbf{F} + \mathbf{R},$$

where $\mathbf{R}$ is the reaction of the curve. This is equivalent to the three scalar equations

$$\left.\begin{aligned} m\frac{dv}{dt} &= F_1, \\ m\kappa v^2 &= F_2 + R_2, \\ 0 &= F_3 + R_3, \end{aligned}\right\}$$

the suffixes 1, 2, 3 denoting resolutes in the directions of the tangent, principal normal and binormal respectively.

SYSTEM OF PARTICLES.

The **linear momentum** of a system of particles is defined as the vector sum of the linear momenta of the separate particles : that is

$$\mathbf{M} = \Sigma m\mathbf{v},$$

which is equivalent to

$$\mathbf{M} = M\bar{\mathbf{v}},$$

where $M = \Sigma m$ is the total mass of the particles, and $\bar{\mathbf{v}}$ the velocity of the centre of mass. The rate of increase of the linear momentum is

$$\frac{d\mathbf{M}}{dt} = M\frac{d\bar{\mathbf{v}}}{dt} = M\bar{\mathbf{a}}.$$

The vector sum of the forces acting on the particles is

$$\Sigma\mathbf{F} = \frac{d}{dt}\Sigma m\mathbf{v} = M\bar{\mathbf{a}},$$

an equation which determines the motion of the c.m.

The **angular momentum** $\mathbf{H}$ of the system about any point is the vector sum of the A.M. of the separate particles. Hence for moments about the origin

$$\mathbf{H} = \Sigma\mathbf{r}\times m\mathbf{v},$$

and its rate of increase is

$$\frac{d\mathbf{H}}{dt} = \Sigma\mathbf{r}\times\frac{d}{dt}(m\mathbf{v}) = \Sigma\mathbf{r}\times\mathbf{F}.$$

Thus the rate of increase of the A.M. about O is equal to the vector sum of the torques about O of all the forces on the system.

This principle may be used for taking moments about the c.m., regarding that point as fixed.

The vector sum of the **impulsive forces** acting on the system is equal to the increase in the linear momentum of the system; that is
$$\Sigma \mathbf{I} = \mathbf{M} - \mathbf{M}_0.$$
The vector sum of the moments of the impulsive forces about a fixed point O is equal to the increase in the A.M. about that point produced by those forces. Thus
$$\Sigma \mathbf{r} \times \mathbf{I} = \mathbf{H} - \mathbf{H}_0.$$

RIGID KINEMATICS.

The motion of a rigid body *about a fixed point* is at any instant one of rotation about a definite axis through that point, called the *instantaneous axis*. The **angular velocity** can then be represented by a vector $\mathbf{A}$ parallel to this axis. The velocity of the particle at the point $\mathbf{r}$ is $\mathbf{v} = \mathbf{A} \times \mathbf{r}$,

the fixed point being taken as origin.

When *no point of the body is fixed*, take the position of any particle as origin and let $\mathbf{v}$ be the velocity of that particle. Then the velocity of any other particle whose position vector is $\mathbf{r}$ is
$$\mathbf{V} = \mathbf{v} + \mathbf{A} \times \mathbf{r}.$$
The vector $\mathbf{A}$ is independent of the origin, and is called the *angular velocity* of the body.

Any motion of a rigid body is equivalent to a **screw motion**. The axis of the screw is parallel to $\mathbf{A}$; and the velocity of any particle on the axis is along the axis, being the same for all such particles. The two *invariants* of the motion are $\mathbf{A}^2$ and $\Gamma = \mathbf{v} \cdot \mathbf{A}$, where $\mathbf{v}$ is the velocity of any particle. The *pitch* of the screw is
$$p = \Gamma / \mathbf{A}^2.$$

Simultaneous angular velocities about a fixed point are compounded by vector addition. Simultaneous angular velocities about parallel axes are compounded like parallel forces. Any **simultaneous motions** corresponding to velocities $\mathbf{v}_1, \mathbf{v}_2, \ldots$ of a particle chosen as origin, and angular velocities $\mathbf{A}_1, \mathbf{A}_2, \ldots$ of the body about that point, are compounded by vector addition of the velocities of the origin, and vector addition of the angular velocities.

RIGID DYNAMICS.

For a body **moving about a fixed point** O, let

$$\mathbf{A} = \omega_1\mathbf{i} + \omega_2\mathbf{j} + \omega_3\mathbf{k}$$

be the angular velocity. Then the *angular momentum* of the body about the fixed point is

$$\mathbf{H} = h_1\mathbf{i} + h_2\mathbf{j} + h_3\mathbf{k},$$

where
$$\left.\begin{aligned} h_1 &= A\omega_1 - F\omega_2 - E\omega_3, \\ h_2 &= B\omega_2 - D\omega_3 - F\omega_1, \\ h_3 &= C\omega_3 - E\omega_1 - D\omega_2, \end{aligned}\right\}$$

A, B, C, D, E, F being the moments and products of inertia of the body with respect to coordinate axes through O parallel to $\mathbf{i}, \mathbf{j}, \mathbf{k}$. There are three mutually perpendicular axes through O for which the products of inertia vanish. These are called the *principal axes* at O; and the corresponding values of A, B, C are the *principal moments of inertia*. An angular velocity about any one of these axes makes $\mathbf{H}$ parallel to that axis.

The *kinetic energy* of the body is

$$T = \tfrac{1}{2}I\omega^2 = \tfrac{1}{2}I\mathbf{A}^2,$$

where $\omega = \text{mod.}\,\mathbf{A}$ and I is the moment of inertia about the instantaneous axis of rotation. Also

$$T = \tfrac{1}{2}\mathbf{A}\cdot\mathbf{H}.$$

The moment of inertia about the axis whose direction cosines are l, m, n relative to the coordinate axes is

$$I = Al^2 + Bm^2 + Cn^2 - 2Dmn - 2Enl - 2Flm.$$

If the body is moving with **no point fixed**, let $\bar{\mathbf{r}}, \bar{\mathbf{v}}$ be the position vector and velocity of the c.m., and $\mathbf{r}', \mathbf{v}'$ those of a particle relative to the c.m. Then the *angular momentum* of the body about the origin is $\quad \mathbf{H} = \bar{\mathbf{r}}\times M\bar{\mathbf{v}} + \Sigma\mathbf{r}'\times m\mathbf{v}'.$

Similarly the *kinetic energy* of the body is

$$T = \tfrac{1}{2}M\bar{\mathbf{v}}^2 + \tfrac{1}{2}\Sigma m\mathbf{v}'^2.$$

The first term is the kinetic energy of translation of the whole body with the velocity of the c.m.; and the second is the energy of the motion relative to the c.m.

The rate of increase of the kinetic energy is equal to the *activity* of all the external forces acting on the body.

Let S_1, S_2 be two **frames of reference in relative motion** about a common fixed point, the motion of S_2 relative to S_1 being an angular velocity $\mathbf{A}$ about that point. Then the rates of change of a vector $\mathbf{r}$ relative to the two frames are connected by

$$\left(\frac{d\mathbf{r}}{dt}\right)_1 = \left(\frac{d\mathbf{r}}{dt}\right)_2 + \mathbf{A}\times\mathbf{r}.$$

This formula is reciprocal, since $-\mathbf{A}$ is the A.V. of S_1 relative to S_2.

Applying the formula to the vector $\mathbf{H}$ representing the A.M. of a body moving about a fixed point O, we obtain **Euler's dynamical equations**. For the frame S_1 we take the system of surrounding objects, at rest relative to each other. The frame S_2 we take as fixed in the moving body. Let $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ be unit vectors, fixed relative to S_2, and parallel to the principal axes of the body at O. The angular velocity

$$\mathbf{A} = \omega_1\mathbf{i} + \omega_2\mathbf{j} + \omega_3\mathbf{k}$$

of the body is also the A.V. of S_2 relative to S_1. If

$$\mathbf{L} = L_1\mathbf{i} + L_2\mathbf{j} + L_3\mathbf{k}$$

is the torque of the external forces about O, the principle of A.M. states that

$$\mathbf{L} = \left(\frac{d\mathbf{H}}{dt}\right)_1 = \left(\frac{d\mathbf{H}}{dt}\right)_2 + \mathbf{A}\times\mathbf{H}.$$

This is the vector equivalent of Euler's three scalar equations

$$\left.\begin{aligned} A\dot{\omega}_1 - (B - C)\omega_2\omega_3 &= L_1, \\ B\dot{\omega}_2 - (C - A)\omega_3\omega_1 &= L_2, \\ C\dot{\omega}_3 - (A - B)\omega_1\omega_2 &= L_3. \end{aligned}\right\}$$

For **Coriolis' Theorem**, connecting the accelerations of a moving point relative to the frames S_1 and S_2, see Art. 95.

RIGID STATICS.

The necessary and sufficient **conditions of equilibrium** for a body acted on by forces $\mathbf{F}_1, \mathbf{F}_2, \ldots$ through the points $\mathbf{r}_1, \mathbf{r}_2, \ldots$ are

$$\left.\begin{aligned} \Sigma\mathbf{F} &= 0, \\ \Sigma\mathbf{r}\times\mathbf{F} &= 0, \end{aligned}\right\}$$

i.e. the vector sum of the forces, and their torque about the origin, must both vanish. If the conditions are satisfied for one origin, they are satisfied for any origin.

A system of forces acting on a body is statically equivalent to any other system having the same vector sum, and the same torque about a common origin.

A system of **parallel forces** $p_1\mathbf{a}$, $p_2\mathbf{a}$, ... acting through the points $\mathbf{r}_1, \mathbf{r}_2, \ldots$, is equivalent to a single force $(p_1+p_2+\ldots)\mathbf{a}$ acting through the point

$$\bar{\mathbf{r}} = \frac{\Sigma p\mathbf{r}}{\Sigma p}.$$

In particular, the *centre of gravity* of the body coincides with its c.m. The last formula ceases to be valid when $\Sigma p = 0$, when the system is equivalent to a couple.

A pair of equal and opposite parallel forces with different lines of action constitutes a **couple.** The torque $\mathbf{L}$ of the system is the same for all origins. It is perpendicular to the plane containing the two lines of action, and equal to

$$\mathbf{L} = (\mathbf{r}_1 - \mathbf{r}_2)\times\mathbf{F},$$

where $\mathbf{r}_1$, $\mathbf{r}_2$ are points on the lines of $\mathbf{F}$ and $-\mathbf{F}$ respectively. Two couples are statically equivalent if they have equal torques. A system of couples is equivalent to a single couple, whose torque is equal to the vector sum of the torques of the individual couples. That is to say, *couples are compounded by vector addition of their torques.*

A system of forces $\mathbf{F}_1, \mathbf{F}_2, \ldots$ through the points $\mathbf{r}_1, \mathbf{r}_2, \ldots$ may be replaced by a single force

$$\mathbf{R} = \Sigma\mathbf{F}$$

through the origin, and a couple of torque

$$\mathbf{G} = \Sigma\mathbf{r}\times\mathbf{F}.$$

The couple varies with the origin, but $\mathbf{R}$ is invariant. The scalar product

$$\Gamma = \mathbf{R}\cdot\mathbf{G}$$

is also an *invariant*. If the origin is on a certain straight line, called the **central axis**, $\mathbf{G}$ is parallel to $\mathbf{R}$. The force and couple then constitute a *wrench*, equivalent to the original system of forces. The *pitch* of the wrench is

$$p = \Gamma/\mathbf{R}^2.$$

Any system of forces acting on a body is statically equivalent to two forces, of which the line of action of one may be chosen arbitrarily. Such a pair are called **conjugate forces** of the system.

If a body, in equilibrium under any set of forces, is supposed started off with any finite motion, the initial rate of work of the forces on the system is zero. Conversely, if the forces are such that, however the body is set in motion, the initial activity of the forces is zero, then the body is in equilibrium under the forces. This is the **principle of virtual work**, or virtual velocities, for the body.

If a **string** is **in equilibrium** under a force $\mathbf{F}$ per unit length, varying from point to point, the tension T is found from the equation

$$\frac{dT}{ds}\mathbf{t} + T\kappa\mathbf{n} + \mathbf{F} = 0,$$

which is equivalent to the scalar equations

$$\left.\begin{aligned} \frac{dT}{ds} + F_1 = 0, \\ \kappa T + F_2 = 0, \end{aligned}\right\}$$

where F_1, F_2 are the resolutes of $\mathbf{F}$ along the tangent and principal normal respectively.

In the case of a **wire or thin rod**, if $\mathbf{S}$, $\mathbf{L}$ represent the stress and bending couple at any section, and $\mathbf{F}$, $\mathbf{G}$ the impressed force and torque per unit length, the equations of equilibrium are

$$\left.\begin{aligned} \frac{d\mathbf{S}}{ds} + \mathbf{F} = 0, \\ \frac{d\mathbf{L}}{ds} + \mathbf{t}\times\mathbf{S} + \mathbf{G} = 0, \end{aligned}\right\}$$

which may be replaced by six Cartesian equations.

ANSWERS TO EXERCISES.

CHAPTER I.

1. Sum $=10\mathbf{i}$. Modules are $\sqrt{74}$, $3\sqrt{10}$, $2\sqrt{46}$. Direction cosines
$$\frac{3}{\sqrt{74}},\ \frac{7}{\sqrt{74}},\ \frac{-4}{\sqrt{74}};\quad \frac{1}{3\sqrt{10}},\ \frac{-5}{3\sqrt{10}},\ \frac{-8}{3\sqrt{10}};\ \text{and}\ \frac{3}{\sqrt{46}},\ \frac{-1}{\sqrt{46}},\ \frac{6}{\sqrt{46}}.$$

2. $4\mathbf{i}-5\mathbf{j}+11\mathbf{k}$. Module $=9\sqrt{2}$. Direction cosines $\dfrac{4}{9\sqrt{2}},\ \dfrac{-5}{9\sqrt{2}},\ \dfrac{11}{9\sqrt{2}}$.

3. $(a_1-b_1)\mathbf{i}+(a_2-b_2)\mathbf{j}+(a_3-b_3)\mathbf{k}$, etc.
Lengths are $\sqrt{\{(a_1-b_1)^2+(a_2-b_2)^2+(a_3-b_3)^2\}}$, etc.

4. $\mathbf{a}+3\mathbf{b},\ 3\mathbf{b}-\mathbf{a},\ 2(\mathbf{a}+\mathbf{b}),\ -(\mathbf{a}+3\mathbf{b})$.

5. $\mathbf{b}-\mathbf{a},\ -\mathbf{a},\ -\mathbf{b},\ \mathbf{a}-\mathbf{b}$.

6. $\frac{1}{2}(\sqrt{3}\mathbf{i}+\mathbf{j}),\ \mathbf{j},\ \frac{1}{2}(\mathbf{j}-\sqrt{3}\mathbf{i}),\ -\frac{1}{2}(\sqrt{3}\mathbf{i}+\mathbf{j}),\ \dfrac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{j}),\ \dfrac{1}{\sqrt{2}}(\mathbf{j}-\mathbf{i})$.

7. $\dfrac{\pi}{6\sqrt{2}}(\mathbf{j}-\mathbf{i}),\ -\dfrac{\pi}{6}\mathbf{i},\ \dfrac{\pi}{12}(\mathbf{i}-\sqrt{3}\mathbf{j})$.

8. $\sqrt{17}$ miles an hour at $\tan^{-1}\frac{1}{4}$ N. of E.

9. $\sqrt{2+2\sin\theta}$ at an angle $\tan^{-1}\left(\dfrac{-\cos\theta}{1+\sin\theta}\right)$ with the fixed diameter.

10. $6\frac{1}{4}$ ft./sec.; 9 ft.; after $1\frac{11}{16}$ sec.

11. Twice the vector determined by the diagonal of the cube drawn from that corner.

12. 5 lb. wt. A force represented by $\dfrac{1}{\sqrt{2}}(5\mathbf{i}+4\mathbf{j}+3\mathbf{k})$.

13. $\sqrt{3}P$ and $2P$ lb. wt.

14. $\sqrt{2}$ times the former, inclined at 135° to it.

16. The point whose distances from the faces through A are $1\frac{3}{4},\ \frac{1}{2},\ 1\frac{1}{4}$.

17. $\dfrac{n+1}{6}(\mathbf{i}+\mathbf{j}+\mathbf{k})$.

18. A force represented by $6\overrightarrow{AO}$, where O is the centroid of the hexagon.

22. The position vectors of the points are $3\mathbf{b}-2\mathbf{a}$ and $2\mathbf{a}-\mathbf{b}$ respectively.

23. The straight line passing through C and the mid-point of AB.

24. On the straight line passing through the mid-point N of the join of the vacant vertices, and the centre O, such that $NO:OG=n-2:2$.

CHAPTER II.

3. $\mathbf{r}=(\mathbf{i}-2\mathbf{j}+\mathbf{k})+t(\mathbf{i}-2\mathbf{k})$; $\frac{1}{4}(6\mathbf{i}-10\mathbf{j}+3\mathbf{k})$.

CHAPTER III.

5. $\frac{1}{\sqrt{155}}(5\mathbf{j}+11\mathbf{k}-3\mathbf{i})$; $\sqrt{\frac{155}{156}}$.

6. $11\mathbf{j}+9\mathbf{k}-3\mathbf{i}$.

8. $\mathbf{r}=\mathbf{d}+t\left(\frac{\mathbf{a}}{a}+\frac{\mathbf{b}}{b}+\frac{\mathbf{c}}{c}\right)$.

12. $(\mathbf{i}+\mathbf{j}-2\mathbf{k})+s(\mathbf{i}-2\mathbf{j}+\mathbf{k})$; $(\mathbf{i}+\mathbf{j}-2\mathbf{k})+s(\mathbf{j}-2\mathbf{i}+\mathbf{k})$, where s is a root of the equation $s^2+2s-2=0$.

15. $\cos^{-1}\frac{1}{3}$; $\cos^{-1}\frac{1}{\sqrt{3}}$.

16. $\mathbf{r}\cdot(3\mathbf{i}-4\mathbf{j}+7\mathbf{k})+13=0$.

17. $\mathbf{r}\cdot(q\mathbf{a}-p\mathbf{b})=0$.

19. $\frac{1}{172}(106\mathbf{i}+73\mathbf{j}-23\mathbf{k})+t(2\mathbf{i}-7\mathbf{j}-13\mathbf{k})$.

20. $\mathbf{r}\cdot(2\mathbf{i}-7\mathbf{j}-13\mathbf{k})=1$.

21. $\frac{1}{3}\sqrt{6}$.

23. $\frac{a}{6}(3-\sqrt{3})(\mathbf{i}+\mathbf{j}+\mathbf{k})$.

26. A plane perpendicular to the straight line joining the two given points.

27. A straight line parallel to the vector difference of the forces, through the intersection of their lines of action.

28. $\sqrt{3}P$ perpendicular to BC. At a point D such that $BD=\frac{1}{3}BC$.

29. 40 units.

34. A sphere.

35. i. A sphere, ii. A plane.

CHAPTER IV.

2. $(\mathbf{a}\cdot\mathbf{d}\,\mathbf{c}\cdot\mathbf{e}-\mathbf{c}\cdot\mathbf{d}\,\mathbf{a}\cdot\mathbf{e})\mathbf{b}+\mathbf{a}\cdot\mathbf{b}(\mathbf{c}\cdot\mathbf{d}\mathbf{e}-\mathbf{c}\cdot\mathbf{e}\mathbf{d})$.

5. $\mathbf{r}\cdot\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=0$.

6. $\mathbf{r}\cdot(\mathbf{a}'-\mathbf{a})\times\mathbf{b}=[\mathbf{a}\mathbf{a}'\mathbf{b}]$.

7. $\mathbf{r}\cdot\mathbf{b}\times\mathbf{c}=[\mathbf{a}\mathbf{b}\mathbf{c}]$.

11. $[\mathbf{a}\mathbf{b}\mathbf{k}]/\text{mod}\,(\mathbf{k}\times\mathbf{b})$. The line of intersection of

$$[\mathbf{r},\ \mathbf{k},\ \mathbf{k}\times\mathbf{b}]=0 \quad\text{and}\quad [\mathbf{r}-\mathbf{a},\ \mathbf{b},\ \mathbf{k}\times\mathbf{b}]=0.$$

13. $\mathbf{r}-\mathbf{c}=t[\mathbf{b}\times(\mathbf{a}-\mathbf{c})]\times[\mathbf{b}'\times(\mathbf{a}'-\mathbf{c})]$.

14. $\mathbf{r}-\mathbf{c}=t\mathbf{a}\times[\mathbf{b}\times(\mathbf{c}-\mathbf{a}')]$.

17. The straight line $\frac{1}{n_1}(q_1-\mathbf{r}\cdot\mathbf{n}_1)=\frac{1}{n_2}(q_2-\mathbf{r}\cdot\mathbf{n}_2)=\frac{1}{n_3}(q_3-\mathbf{r}\cdot\mathbf{n}_3)$.

22. $\frac{1}{N}(\mathbf{d}\times\mathbf{d}')$, $\frac{1}{N^2}\{(\mathbf{d}\times\mathbf{d}')\times\mathbf{d}'[\mathbf{m}\mathbf{d}\mathbf{d}']+(\mathbf{d}\times\mathbf{d}')\times\mathbf{d}[\mathbf{m}'\mathbf{d}'\mathbf{d}]\}$, where $N=\text{mod}\,\mathbf{d}\times\mathbf{d}'$.

CHAPTER V.

1. i. $2r\dot{r}\mathbf{r}+r^2\dot{\mathbf{r}}+\mathbf{a}\cdot\dot{\mathbf{r}}\mathbf{b}$.

   ii. $3r^2\dot{r}\mathbf{r}+r^3\dot{\mathbf{r}}+\mathbf{a}\times\ddot{\mathbf{r}}$.

   iii. $2(a\mathbf{r}+r\mathbf{b})\cdot(a\dot{\mathbf{r}}+\dot{r}\mathbf{b})$.

   iv. $\frac{\dot{\mathbf{r}}}{r^3}-\frac{2\dot{r}\mathbf{r}}{r^3}+\frac{\mathbf{b}}{\mathbf{a}\cdot\mathbf{r}}-\frac{r(\mathbf{a}\cdot\dot{\mathbf{r}})\mathbf{b}}{(\mathbf{a}\cdot\mathbf{r})^2}$.

   v. $2r\dot{r}-\frac{2\dot{r}}{r^3}$.

   vi. $m\dot{\mathbf{r}}\cdot\ddot{\mathbf{r}}$.

2. First derivatives are

$$\left[\mathbf{r}\,\frac{d\mathbf{r}}{dt}\,\frac{d^3\mathbf{r}}{dt^3}\right] \quad\text{and}\quad \frac{d\mathbf{r}}{dt}\times\left(\frac{d\mathbf{r}}{dt}\times\frac{d^2\mathbf{r}}{dt^2}\right)+\mathbf{r}\times\left(\frac{d\mathbf{r}}{dt}\times\frac{d^3\mathbf{r}}{dt^3}\right).$$

Second derivatives are $\left[\mathbf{r}\,\frac{d^2\mathbf{r}}{dt^2}\,\frac{d^3\mathbf{r}}{dt^3}\right]+\left[\mathbf{r}\,\frac{d\mathbf{r}}{dt}\,\frac{d^4\mathbf{r}}{dt^4}\right]$, and another expression.

3. $\dfrac{\dot{\mathbf{r}}}{\mathbf{r}^2+\mathbf{a}^2} - \dfrac{2r\dot{r}(\mathbf{r}+\mathbf{a})}{(\mathbf{r}^2+\mathbf{a}^2)^2}$, $\dfrac{\dot{\mathbf{r}}\times\mathbf{a}}{\mathbf{r}\cdot\mathbf{a}} - \dfrac{\dot{\mathbf{r}}\cdot\mathbf{a}\,\mathbf{r}\times\mathbf{a}}{(\mathbf{r}\cdot\mathbf{a})^2}$.

5. i $\mathbf{r}=\frac{1}{2}\mathbf{a}t^2+\mathbf{b}t+\mathbf{c}$, where $\mathbf{b}$ and $\mathbf{c}$ are const.

   ii. $\mathbf{a}\times\mathbf{r}=\frac{1}{2}\mathbf{b}t^2+\mathbf{c}t+\mathbf{d}$, where $\mathbf{c}$ and $\mathbf{d}$ are const.

6. $\mathbf{r}=\frac{1}{6}\mathbf{a}t^3+\frac{1}{2}\mathbf{b}t^2$.

7. i. Mod $\mathbf{r}$ is constant. ii. Direction of $\mathbf{r}$ is constant.

8. i. $\left[\dfrac{d\mathbf{r}}{ds}\,\dfrac{d^2\mathbf{r}}{ds^2}\,\dfrac{d^3\mathbf{r}}{ds^3}\right]=0.$ 11. $\alpha=\dfrac{\pi}{4}$.

13. $\kappa=\dfrac{a}{6(a^2+c^2)\sin 2\theta}$. $\mathbf{n}=\sin\theta\,.\,\mathbf{i}+\cos\theta\,.\,\mathbf{j}$.

   Hence the equations of the principal normal and the plane of curvature may be written down.

23. $R=a\sec^2\alpha$. 25. $R=3a(1+t^2)^2\sqrt{1+8t^2}$.

CHAPTER VI.

1. $\frac{1}{2}\{\mathbf{r}_1\times(\mathbf{v}_2-\mathbf{v}_3)+\mathbf{r}_2\times(\mathbf{v}_3-\mathbf{v}_1)+\mathbf{r}_3\times(\mathbf{v}_1-\mathbf{v}_2)\}$.

2. $\dfrac{1}{6}\dfrac{d}{dt}\{[\mathbf{r}_1\mathbf{r}_2\mathbf{r}_3]-[\mathbf{r}_1\mathbf{r}_2\mathbf{r}_4]+[\mathbf{r}_1\mathbf{r}_3\mathbf{r}_4]-[\mathbf{r}_2\mathbf{r}_3\mathbf{r}_4]\}$, in which $\dfrac{d\mathbf{r}_1}{dt}=\mathbf{v}_1$, etc.

13. The new major axis is double the original one.

25. The speed is $\sqrt{2gz}$. The resolutes of the reaction along the principal normal and the binormal are $\dfrac{2mgz}{a}\cos^2\alpha$ and $-mg\cos\alpha$ respectively.

CHAPTER VII.

4. The instantaneous centre of rotation is the point of intersection of the perpendiculars to the two straight lines at the extremities of the rod. In the second case the rotation is round an axis perpendicular to the two given straight lines.

17. If θ is the inclination (to the vertical) of the radius to the bead

$$v^2=2ga(1-\cos\theta)+\omega^2a^2\sin\theta(2+\sin\theta),$$

and if x is the distance of the bead from the fixed vertical tangent, the components R, R' of the reaction on the bead, along the radius and perpendicular to the plane of the wire, are given by

$$\frac{v^2}{a}=g\cos\theta-\omega^2x\sin\theta+\frac{R}{m},$$

$$\frac{1}{x}\frac{d}{dt}(x^2\omega)=\frac{R'}{m}.$$

$$\dot{r}-r\dot{\theta}^2-r\sin^2\theta\dot{\phi}^2,$$

$$\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})-r\sin\theta\cos\theta\dot{\phi}^2,$$

$$\frac{1}{r\sin\theta}\frac{d}{dt}(r^2\sin^2\theta\dot{\phi}).$$

CHAPTER VIII.

81. If R is the reaction per unit length,

$$\frac{dT}{ds} = \mu R + \rho g \sin \psi,$$

$$\kappa T = R - \rho g \cos \psi,$$

κ being the curvature at the point, ρ the mass per unit length, and ψ the inclination of the tangent to the horizontal.

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